Mathematical Reviews

June, 1940 TABLE OF CONTENTS Vol. 1, No. 6

THE REPRINT SERVICE

The subscribers to MATHEMATICAL REVIEWS can obtain, either on microfilm or as photoprint, the complete text of any article of which an abstract is printed, except books, or material of which the reproduction is prohibited by copyright law. The microfilm requires some sort of reading machine or projector in which standard 35 mm. double perforated film can be viewed. Only *positive* prints are furnished; that is, the projected image appears as black words on a white background. Photoprints, on the other hand, are on sheets 10" by 14", and are similar to photostats. They can be read directly without the aid of any reading device. They also are furnished as positives, that is, black words on a white background.

Cost

The cost of microfilm is 2¢ per exposure, *each exposure* (except in the case of a very few unusually large journals) *containing two pages of text.*

Photoprints cost 16¢ per sheet, *each sheet* (except in the case of unusually large journals) *containing two pages of text.*

The minimum charge for any one order (but not for any one item when several items are ordered simultaneously) is 50¢.

How to Order

The process of ordering either microfilm or photoprint may be made clear by considering an example. Each abstract bears a serial number easily recognized by the fact that it is always preceded by the abbreviation "MF". A subscriber who wishes the text of item 18 would first note that the original article occupied pages 590 to 608. It therefore covers 19 pages, and will require ten exposures of microfilm, or ten sheets of photoprint for its reproduction. Hence the microfilm charge for this particular item would be 20¢, and the photoprint charge $1.60. The charges so computed apply on any order totaling 50¢ or more; if the total is less than 50¢, the charge for the entire order will be 50¢.

In ordering:

1. Give your name and address.
2. State the serial number of the article or articles you wish.
3. Say whether you wish microfilm or photoprint.
4. Enclose cash for the amount of the order, either in currency, stamps, or by check.
5. Mail to MATHEMATICAL REVIEWS, Brown University, Providence, Rhode Island.

In the case of large journals, only one page of which can be accommodated on a microfilm exposure or a photoprint sheet, you will be notified of the extra charge before the order is filled, unless you have indicated on your order that the reprint is desired even at the higher price.

Why Cash Must Accompany Order

Individual orders for either microfilm or photoprint are so small that if credit books were kept and bills rendered, the extra clerical expenses would be a large proportion of the total. By requiring payment in advance, these charges have been eliminated, and the price to the customer correspondingly reduced. However, libraries and institutions which make extensive use of the service may be granted the privilege of payment at stated intervals.

MATHEMATICAL REVIEWS

Published monthly by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
ACADEMIA NACIONAL DE CIENCIAS EXACTAS, FISICAS Y NATURALES DE LIMA
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879.

Mathematical Reviews

Vol. 1, No. 6 JUNE, 1940 Pages 161-192

THEORY OF GROUPS

Miller, G. A. Groups which contain less than fourteen proper subgroups. Proc. Nat. Acad. Sci. U. S. A. **26**, 129–132 (1940). [MF 1282]

Continuing a series of papers classifying groups according to the number of proper subgroups, the author lists all groups which separately contain twelve or thirteen proper subgroups. Of the former there are fifteen types including six abelian, and of the latter there are eight types including four abelian. *J. S. Frame* (Providence, R. I.).

Piccard, Sophie. Quelques propositions concernant les bases du groupe symétrique et du groupe alternant. Comment. Math. Helv. **12**, 130–148 (1939–40). [MF 1046]

Two permutations S, T of the symbols $1, 2, \cdots, n$ are said to be primitive in this paper if the group generated by S, T is transitive on the n symbols and primitive in the ordinary sense. Since the symmetric (alternating) group $\mathfrak{S}_n(\mathfrak{A}_n)$ is a primitive group, a necessary condition for S, T to generate $\mathfrak{S}_n(\mathfrak{A}_n)$ is that S, T be primitive; but this is not a sufficient condition. Theorems 1–13, giving necessary and sufficient conditions that special forms of S, T should generate $\mathfrak{S}_n(\mathfrak{A}_n)$, are stated without proof. For $r=6$, theorem 14 is given by: For any integer $n\geqq 2r-1$ ($r=2, 3, 4, 5, 6$) and r symbols a_i, chosen from $1, 2, \cdots, n$, such that

$$(1)\qquad 1\leqq a_1<a_2<\cdots<a_r\leqq n,$$

a necessary and sufficient condition that $S=(12\cdots n)$, $T=(a_1a_2\cdots a_n)$ should generate $\mathfrak{S}_n(\mathfrak{A}_n$, if n, r are both odd) is that S, T be primitive. By dropping (1), we obtain theorems 1, 6 for $r=2, 4$ and weakened forms of theorems 3, 13 for $r=3, 5$. Theorem 16 states that for any r there is an n such that S, T generate $\mathfrak{S}_n(\mathfrak{A}_n$, if n, r are both odd), again without (1). Naturally one asks, is the condition $n\geqq 2r-1$ significant in general, with or without (1)?

G. deB. Robinson (Toronto, Ont.).

Frame, J. S. On the decomposition of transitive permutation groups generated by the symmetric group. Proc. Nat. Acad. Sci. U. S. A. **26**, 132–139 (1940). [MF 1283]

If G be the symmetric group on n symbols and H a subgroup of G, we may represent the permutation group on the cosets HS_i, written as a group of permutation matrices, by the symbol G_H. In particular, if $H=H_\alpha$ is taken to be the direct product of symmetric groups on $\alpha_1, \alpha_2, \cdots, \alpha_n$ symbols, where $\sum\alpha_i=n$, then the reduction of the corresponding permutation group $G_H(\alpha)$ into its irreducible components $\{\gamma\}$ may be represented by the equation

$$G_H(\alpha)=\sum_\gamma \mu_\gamma{}^\alpha\{\gamma\},$$

where $\mu_\gamma{}^\alpha$ is the multiplicity with which $\{\gamma\}$ appears in $G_H(\alpha)$. Frobenius proved [S.-B. Berlin Math. Ges. **36**, 501–515 (1898)] that the number of double cosets $H_\alpha SH_\alpha$ contained in G is given by $\sum_\gamma(\mu_\gamma{}^\alpha)^2$. Extending a result of Robinson [Amer. J. Math. **60**, 745–760 (1938)], the author proves that these double cosets $H_\alpha SH_\alpha$ may be displayed in a series of squares, one for each of the irreducible components $\{\gamma\}$ of $G_H(\alpha)$, so that each square contains $(\mu_\gamma{}^\alpha)^2$ double cosets which are arranged so that inverse double cosets are symmetrically placed with respect to the diagonal and the self-inverse double cosets occupy the diagonal positions. Consequently, the number of self-inverse double cosets $H_\alpha SH_\alpha$ in G is equal to $\sum_\gamma\mu_\gamma{}^\alpha$.

G. deB. Robinson (Toronto, Ont.).

Dubuque, P. Une généralisation des théorèmes de Frobenius et Weisner. Rec. Math. (Moscou) **5** (47), 189–196 (1939). (Russian. French summary) [MF 1429]

Four theorems are proved about the number of elements X of a finite group $\mathfrak{G}$ such that (I) the nth power of X lies in a class $\mathfrak{A}$ of conjugate elements of order a, and (II) some power of X lies in a given system $\mathfrak{M}$ of elements of order m. It is assumed that m is a factor of n, and that n is a factor of the order g of the group. The proof depends on factoring n into relatively prime factors $n=l'm'a_1r$, where the prime factors of l', m', a_1 and r are prime factors, respectively, of neither a nor m, m but not a, a but not m, both a and m. Theorem 1 assumes (II_1) that $\mathfrak{M}$ is a class of conjugate elements of order m, and states that the number of elements X satisfying (I), (II) and (II_1) is a multiple of $l'a_1$, the greatest divisor of n which is prime to m. As special cases, this theorem includes, for $m=1$, a theorem of Frobenius [S.-B. Preuss. Akad. Wiss. **1903**, 987] and, for $a=1$, $n=g$, a theorem of M. L. Weisner [Bull. Amer. Math. Soc. **31**, 492–496 (1925)]. A theorem of W. K. Turkin [C. R. Acad. Sci. Paris **193**, 1059–1061 (1931)] is readily obtained as a corollary. Theorem 2 assumes (II_2) that $\mathfrak{M}$ is an arbitrary system of elements of order m prime to a, and states that the number of elements X satisfying (I), (II) and (II_2) is a multiple of $\phi(m)$. Theorems 3 and 4 assume (II_3) that $\mathfrak{M}$ consists of all elements of order m, and thus apply theorems 1 and 2 to show that the number of elements X of a group whose order is a multiple of m is divisible by $l'a_1$, and is also divisible by $\phi(m)$ if a is prime to m. *J. S. Frame.*

Suetuna, Zyoiti. Über die Zerlegung der Gruppencharaktere. Jap. J. Math. **16**, 79–91 (1939). [MF 1109]

Using the methods of a previous paper [J. Fac. Sci. Tokyo **3**, 223–252 (1937)], the author treats a simple case in which the factor-group $\mathfrak{G}/\mathfrak{H}$ is insoluble, namely, $\mathfrak{G}/\mathfrak{H}$ is the icosahedral group. *D. E. Rutherford* (St. Andrews).

Tchounikhin, S. A. Einige Sätze über einfache Gruppen. Rec. Math. (Moscou) **5** (47), 537–543 (1939). (German. Russian summary) [MF 1344]

The two main results are as follows. First, let $\mathfrak{P}$ be a Sylow subgroup of order p^b of a simple group of order p^bn, where n is coprime to $p-1$. Suppose the central of $\mathfrak{P}$ to be of order p^ν and type $(\lambda_1, \lambda_2, \cdots, \lambda_r)$, where $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_r$. Define $\omega_1=\lambda_1$ if $r=1$, and otherwise $\omega_1=\lambda_1-\lambda_2$, $\omega_i=\min(\lambda_{i-1}-\lambda_i, \lambda_i-\lambda_{i+1})$ for $1<i<r$, and $\omega_r=\min(\lambda_{r-1}-\lambda_r, \lambda_r)$. Then

$\omega_i \leqq \delta - \eta$ for all i. Secondly, if the Sylow subgroup $\mathfrak{P}$ is Abelian, of type $(\lambda_1, \lambda_2, \cdots, \lambda_r)$, and if ρ is the smallest number for which $(p^\rho - 1, n) > 1$, then the λ's are equal in sets of ρ or more, and therefore $\lambda_1 \leqq \delta/\rho$, that is, $\mathfrak{P}$ contains no element of order greater than $p^{\delta/\rho}$.

H. S. M. Coxeter (Toronto, Ont.).

Yamada, Kaneo. Ein Kriterium für die Nichteinfachheit der Gruppen. Tôhoku Math. J. **46**, 44–45 (1939). [MF 1166]

To obtain conditions that a group $\mathfrak{G}$ be not simple, the author considers the permutation group $\mathfrak{G}^*$ of degree n on the cosets with respect to a subgroup H of index n. If $\mathfrak{G}$ has an element of order $p^\alpha q^\beta \cdots r^\sigma$ for which $p^\alpha + q^\beta + \cdots + r^\sigma > n$, then $\mathfrak{G}^*$ cannot be one-to-one isomorphic with $\mathfrak{G}$; hence $\mathfrak{G}$ is not simple. If $\mathfrak{G}$ has an element of order n and $\mathfrak{G}^*$ is simply transitive, then $\mathfrak{G}$ is not simple. As a consequence a group of order $p^\alpha q^\beta$ having a cyclic Sylow subgroup is soluble.

J. S. Frame (Providence, R. I.).

Tschernikow, S. Über unendliche spezielle Gruppen. Rec. Math. (Moscou) **6** (48), 199–214 (1939). (Russian. German summary) [MF 1353]

A special group is one (i) whose every sequence of properly decreasing subgroups is of finite "length" and (ii) whose proper subgroups never coincide with their normalizers. By (i) these groups are periodic: all elements of finite order. A periodic group satisfying (ii) is a direct product of its Sylow subgroups: maximal p-groups (every g of order p^m, $m = m(g)$, p a fixed prime). The factor groups of a special group are themselves special; such a group is always the direct product of a finite number of special p-groups. Conversely, products of this sort are always special groups. The theorem that every special group has a non-trivial center is proved, first for special p-groups. The proof is by induction on a transfinitely constructed ascending sequence of normalizers which begins with an arbitrary cyclic subgroup; the simple fact that a special group has at most a finite number of elements of order p plays a decisive role in the induction. Using in addition the theorem on factor groups, it is next shown that every special group G possesses at least one central sequence (transfinite, well-ordered) of normal subgroups N_a: (*) $N_0 = 1 \subset N_1 \subset N_2 \subset \cdots \subset N_a \subset \cdots \subset N_c = G$, where (1) $N_a = \sum_{b<a} N_b$ for every "limit" ordinal a and (2) every element of N_{a+1}/N_a, $a<c$, is invariant in G/N_a. On the other hand, a group possessing a central sequence and satisfying (i) is special. A subgroup of a special group, as well as a finite direct product of special groups, satisfies (i) and has a central sequence: is therefore special.

A group is called quasi-special (q.s.) if it possesses a periodic sequence (having the same form as (*)), where now (1) N_a is a normal divisor of N_{a+1}, $a<c$, (2) N_{a+1}/N_a is of prime order, if these groups are distinct, (3) $N_a = \sum_{b<a} N_b$, if a is a limit ordinal, and (4) the periodic sequence is a principal one: here the N_a are normal divisors of G. A special group is q.s.; a q.s. group need not be special. The subgroups of a q.s. group are q.s.; the commutator and all Sylow subgroups are special. Every q.s. group is solvable and enumerable. Every group possessing a periodic sequence (in particular every special group) is locally finite: every finite set of its elements generates a finite subgroup. A group is special if and only if it is enumerable, locally finite, and satisfies (i). An infinite special group P has a unique minimal subgroup Q of finite index and this group has an infinite center. If, in addition, P is a p-group, then Q possesses a central sequence all of whose terms are normal divisors of P; moreover the factor groups Q_{a+1}/Q_a of this central sequence are the direct product of a finite number of primary abelian groups of type p^∞. Finally, a characterization of these "elementary" groups: P is of type p^∞ if it is an infinite p-group, locally finite, all of whose subgroups are finite (it would seem that the condition of local-finiteness is redundant).

L. Zippin (Flushing, N. Y.).

de Kerékjártó, B. Sur les inversions dans un groupe commutatif. C. R. Acad. Sci. Paris **210**, 288–289 (1940). [MF 1600]

A one-one correspondence f between the elements of the abelian group G is termed an inversion if $f(x) = x^{-1}$ for every x in G, and it is termed a translation if $f(x) = x + a$ for every x in G and for some suitable a in G. If the abelian group G does not contain elements of order 2, then the following two conditions are proved to be necessary and sufficient for a one-one correspondence f to be the inversion: (a) if t is a translation, then $ftft^{-1}$ is a translation; (b) the only fixed-element of f is the identity. This theorem has applications in the theory of topological groups. [Cf. in this context: O. Taussky, Math. Ann. **108**, 615–620 (1933).]

R. Baer (Urbana, Ill.).

Specht, Wilhelm. Zur Theorie der Gruppen linearer Substitutionen. II. Jber. Deutsch. Math. Verein. **49**, 207–215 (1940). [MF 1215]

Consider a semi-group $\mathfrak{G}$ of matrices of degree n, whose elements are complex numbers. By a suitable choice of a unitary matrix U we may write

$$U\mathfrak{G}U^* = \begin{pmatrix} \mathfrak{G}_1 & 0 & \cdots & 0 \\ \mathfrak{G}_{21} & \mathfrak{G}_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \mathfrak{G}_{k1} & \mathfrak{G}_{k2} & \cdots & \mathfrak{G}_k \end{pmatrix},$$

where U^* is the conjugate transpose of U; if $k > 1$, $\mathfrak{G}$ is reducible, otherwise irreducible. If every $\mathfrak{G}_{ij} = 0$, $\mathfrak{G}$ is completely reducible. One criterion for the complete reducibility of $\mathfrak{G}$ is that $\mathfrak{G} = \mathfrak{G}^*$. A basis of the enveloping algebra $\mathfrak{R}(\mathfrak{G})$ of $\mathfrak{G}$ is a set of elements $G_1, G_2, \cdots, G_r$ such that for any element G of $\mathfrak{G}$

$$G = \alpha_1 G_1 + \alpha_2 G_2 + \cdots + \alpha_r G_r,$$

where the α's are complex numbers. If $\mathfrak{G}$ is a group, $\mathfrak{R}(\mathfrak{G})$ is the associated group ring, and the author proves that $\mathfrak{G}$ is completely reducible if $G_i^{*-1}\mathfrak{G}G_i^*$ is contained in $\mathfrak{R}(\mathfrak{G})$ for every i.

G. deB. Robinson (Toronto, Ont.).

Schubarth, Emil. Über normal-diskontinuierliche lineare Gruppen in zwei komplexen Variablen. Comment. Math. Helv. **12**, 81–129 (1939–40). [MF 1045]

Soient $w = w_1/w_3$ et $z = w_2/w_3$ deux variables complexes. Les formules $w_\alpha' = \sum_{\beta=1}^3 a_{\alpha,\beta} w_\beta$ $(\alpha = 1, 2, 3)$, où l'on suppose que le déterminant $\Delta(a_{\alpha,\beta})$ n'est pas nul, définissent une transformation appliquée à (w, z). Ce mémoire étudie les propriétés des groupes de telles transformations au point de vue de la formation des fonctions automorphes (cette expression désigne les fonctions $f(w, z)$ qui sont invariantes par un tel groupe et méromorphes dans un domaine invariant par le groupe). Cette étude, très générale, continue les travaux de plusieurs auteurs, dont elle donne la liste. En un point donné, la discontinuité du groupe consiste en ce que seulement un nombre fini de transformés d'un voisinage de ce point ont des points communs avec ce voisinage. Cette discontinuité ne suffisant pas pour assurer l'existence

des fonctions automorphes, Myrberg introduisit la notion de discontinuité normale, spécialement étudiée ici. Outre celle-ci, l'auteur introduit aussi une notion de semi-discontinuité (bedingte Diskontinuität). En même temps que le groupe donné, il s'est montré avantageux d'étudier aussi le groupe qui lui correspond dualistiquement, ou groupe des transformations subies par des variables ω_1, ω_2, ω_3 telles que $\sum_\alpha \omega_\alpha w_\alpha$ soit invariant; cette étude est présentée en langage géométrique. Un théorème fondamental lie la discontinuité normale du groupe donné en un point donné à la semi-discontinuité du groupe associé en tout élément $(\omega_1, \omega_2, \omega_3)$ tel que $\sum_\alpha \omega_\alpha w_\alpha$ soit nul. Une section importante du travail est consacrée aux images isométriques des transformations du groupe; cette expression désigne l'ensemble des points (w, z) où le déterminant fonctionnel a un pour valeur absolue; de cette notion (introduite par L. R. Ford pour une seule variable) l'auteur tire une méthode propre à déterminer, dans un cas très général, un domaine fondamental du groupe (domaine dans lequel existe un et un seul transformé de tout point d'un certain domaine invariant par le groupe). Une dernière section établit pour certains domaines rencontrés dans le travail des propriétés géométriques analogues à la convexité et esquisse la généralisation pour n variables. Enfin est indiquée la formation des fonctions automorphes; celles-ci existent dans tout domaine où le groupe est normalement discontinu. *G. Giraud.*

Kawada, Yukiyosi. Bemerkungen zur Theorie der allgemeinen Kugelfunktionen. Proc. Imp. Acad., Tokyo **15**, 334–339 (1939). [MF 1182]

Soient $\mathfrak{G}$ un groupe topologique, et $\mathfrak{F}$ une famille de fonctions presque périodiques sur $\mathfrak{G}$, fermée par rapport aux opérations linéaires et de translation à droite. L'auteur étudie les représentations unitaires de $\mathfrak{G}$ qui ont des modules de représentation contenus dans $\mathfrak{F}$. Si $\mathfrak{d}$ est une de ces représentations de degré n, on peut remplacer $\mathfrak{d}$ par une représentation équivalente $s \rightarrow (d_{ij}(s))$, normée par rapport à $\mathfrak{F}$, pour laquelle les fonctions d_{ij} $(1 \leqq i \leqq r, 1 \leqq j \leqq n)$ sont dans $\mathfrak{F}$, r étant un certain entier $(r \leqq n)$. Les combinaisons linéaires des fonctions $d_{ij}(s)$ permettent d'approximer les fonctions sur $\mathfrak{G}$ qui sont dans $\mathfrak{F}$. Si $\mathfrak{H}$ est un sous-groupe de $\mathfrak{G}$, les fonctions presque périodiques f pour lesquelles $f(ts) = f(s)$ pour $t \epsilon \mathfrak{H}$ forment une famille $\mathfrak{F}$. L'invariant r relatif à une représentation $\mathfrak{d}$ et à la famille $\mathfrak{F}$ est le nombre de fois que la représentation identique de $\mathfrak{H}$ est contenue dans la représentation donnée par $\mathfrak{d}$. L'auteur applique ces résultats au cas où $\mathfrak{G}$ est le groupe orthogonal, $\mathfrak{H}$ étant le groupe qui laisse un point de l'espace invariant, et étudie les représentations de $\mathfrak{G}$ fournies par les pôlynomes sur la sphère unitaire. *S. Bochner* (Princeton, N. J.).

Neumer, Walter. Die dreigliedrigen Berührungstransformationsgruppen der Ebene, welche keine Invariante erster Ordnung besitzen. I. J. Reine Angew. Math. **181**, 133–152 (1939). [MF 1417]

This paper is a supplement to a previous paper dealing with the most general group of contact transformations similar to the projective G_8 of the plane [J. Reine Angew. Math. **173** (1935)]. It deals mainly with the three parameter contact transformations G_3 in the plane under which no function I (x, y, y') is invariant. These groups have the property that they leave invariant (but for constant factors) ∞^1 elements of arc. Examples of such G_3 are those similar to the projective group of a conic, and those similar to the group of euclidean motions. Special cases include former results of the author [J. Reine Angew. Math. **176**, 224–249; **177**, 13–36, 65–81]. *D. J. Struik* (Cambridge, Mass.).

Flexner, William W. Character group of a relative homology group. Ann. of Math. **41**, 207–214 (1940). [MF 1011]

Let L be a subcomplex of the finite complex K, and G a discrete or bicompact abelian group. Let $\mathfrak{A}^G$ be the group formed by those elements of the p-G-homology group of L (that is, the p-dimensional homology group with coefficients in G) whose members bound in K. The main result of this paper is the following geometrical interpretation of the character group of $\mathfrak{A}^G$. Let H be the character group of G, $\mathfrak{Z}_H(L)$ the group of p-H-cocycles of L, and let $\mathfrak{Z}_H(K)^*$ consist of the members x of $\mathfrak{Z}_H(L)$ such that x is the part in L of a cocycle of K. Then the factor group $\mathfrak{B}_H = \mathfrak{Z}_H(L)$ mod $\mathfrak{Z}_H(K)^*$ is the character group of $\mathfrak{A}^G$. It is further shown that $\mathfrak{B}_H \approx \mathfrak{H}_H(L)$ mod Δ, where $\mathfrak{H}_H(L)$ is the p-H-cohomology group of L and Δ is its subgroup of cohomology classes that have representatives in $\mathfrak{Z}_H(K)^*$. This is proved by obtaining a second form $\mathfrak{B}_H'$ of the character group of $\mathfrak{A}^G$, and using the theorem $\mathfrak{B}_H \approx \mathfrak{B}_H'$; it also follows directly from Theorem 2.4 (and therefore for any H), since ${}^p\mathfrak{F}_H(L) < \mathfrak{Z}_H(K)^*$. In the first part of the paper the theorems on orthogonal groups required (and some others) are established, on the basis of Pontrjagin's work. Example (Theorem 2.3): let G_1 and G_2 be the direct sums of, respectively, σ and $s-\sigma$ specimens of a group G, and let H_1 and H_2 be similarly derived from H, orthogonal to G. Let F be a subgroup of H_1+H_2 and let F^o and F^* consist of those elements x of H_1 for which, respectively, $x+0 \epsilon F$, and $x+y \epsilon F$ for some y. Then

$$(G_1+G_2, F)^o \approx (G_1+G_2, F^*+H_2),$$

where (X, Y) is the annihilator of Y in X.
M. H. A. Newman (Cambridge, England).

Gantmacher, Felix. Canonical representation of automorphisms of a complex semi-simple Lie group. Rec. Math. (Moscou) **5** (47), 101–146 (1939). (English. Russian summary) [MF 1426]

Let $\mathfrak{R}$ be a complex semi-simple Lie algebra. The problem considered is that of representing automorphisms of $\mathfrak{R}$ in terms of particularly simple automorphisms. Let $h_1, \cdots, h_n, e_\alpha, e_\beta, \cdots$ be a canonical basis for $\mathfrak{R}$. The h's constitute a basis for a maximal nilpotent (necessarily abelian) subalgebra $\mathfrak{h}$ (n being the rank of $\mathfrak{R}$) and the remaining elements correspond to the root-vectors $\alpha, \beta, \cdots$. These vectors may be considered as being in the space $\mathfrak{h}$ and among them there are n linearly independent ones. Let $\mathfrak{A}_0, \mathfrak{A}_1, \cdots$ be the connected components of the group $\mathfrak{A}$ of all automorphisms of $\mathfrak{R}$, $\mathfrak{A}_0$ being the component which contains the unit automorphism and is therefore identical with the group of inner automorphisms. Let $\mathfrak{T}$ be the finite group consisting of those linear transformations of $\mathfrak{h}$ into itself which permute the root vectors among themselves. The reflections across hyperplanes orthogonal to the root vectors generate a normal subgroup $\mathfrak{S}$ of $\mathfrak{T}$ and the index k of $\mathfrak{S}$ equals the number of components $\mathfrak{A}_i$. Let $\tau_0, \cdots, \tau_{k-1}$ be representative elements in the cosets of $\mathfrak{S}$ with $\tau_0 \subset \mathfrak{S}$. It is shown that there exist in $\mathfrak{A}_i$ certain "chief" automorphisms Z of the form

$$(Z-\tau_i)\mathfrak{h}=0, \quad Ze_\alpha = k_\alpha e_{\alpha'}, \quad \alpha' = \tau_i(\alpha), \quad \alpha = \alpha, \beta, \cdots,$$

the k_α being scalars depending, among other things, on the particular canonical basis chosen and on the choice of the τ_i.

In particular, if $i=0$, then $\alpha=\alpha'$, so that the matrices of chief inner automorphisms are diagonal. (The chief inner automorphisms are in fact identical with the automorphisms e^H, where $Hx=[h, x]$, $h\subset\mathfrak{h}$.) The main result can now be stated: if an automorphism A in $\mathfrak{A}_i$ has simple elementary divisors, it is representable in the form UZU^{-1}, where Z is a chief automorphism in $\mathfrak{A}_i$ and U is an inner automorphism. In particular, if A is an inner automorphism, it is sufficient to assume that the elementary divisors corresponding to the characteristic number 1 are simple, for then all elementary divisors are simple. Fundamental lemma: let A be an automorphism of $\mathfrak{R}$ and let $\mathfrak{K}_1$ be the invariant subspace of $\mathfrak{R}$ which corresponds to the characteristic number 1 of A; let $\mathfrak{M}_A$ be the component containing A of the group $\mathfrak{M}$ of all automorphisms commutative with A; then $\mathfrak{K}_1$ is a subalgebra and, if it is abelian, $\mathfrak{M}_A$ is an abelian system and the systems $U\mathfrak{M}_AU^{-1}$, where U runs through a neighborhood of the unit automorphism, cover a neighborhood of A in $\mathfrak{A}$. *P. A. Smith* (New York, N. Y.).

Kowalewski, Gerhard. Zur Cesàro-Pickschen Geometrie. J. Reine Angew. Math. **181**, 218–241 (1940). [MF 1211]

If $\xi_i p+\eta_i q$, $i=1, \cdots, r>2$ are the generators of an r-parameter Lie group in the x, y plane, the elements of the $(r-2)$th extension of this group will be $(x, y, y', y'', \cdots, y^{(r-2)})$. If the generic rank of the matrix of this extended group is r, the group will be simply transitive and the geometry of these $(r-2)$ dimensional elements is what the author calls a Cesàro-Pick geometry. The paper, however, is devoted to the construction of the generators of these extended groups for the 22 types of groups in the plane of three or more parameters. It is also shown that in each case two basic generators X_1f and X_2f can be obtained (the author calls them male and female) so that all the other generators are successive commutators of these two (their children). *M. S. Knebelman* (Pullman, Wash.).

Kawada, Yukiyosi und Kondô, Kôiti. Idealtheorie in nicht kommutativen Halbgruppen. Jap. J. Math. **16**, 37–45 (1939). [MF 1104]

Let $\mathfrak{R}$ be a non-commutative semigroup whose center contains a group $\mathfrak{g}$. A subsemigroup $\mathfrak{O}$ of $\mathfrak{R}$ is called an order (with respect to $\mathfrak{g}$) if every element of $\mathfrak{R}$ can be expressed in the form s/z with s in $\mathfrak{O}$ and z in $\mathfrak{o}=\mathfrak{O}\cap\mathfrak{g}$; $\mathfrak{O}$ is a maximal order if every order $\mathfrak{O}^*\supseteqq\mathfrak{O}$ for which $\lambda\mathfrak{O}^*\subseteqq\mathfrak{O}$, with λ in $\mathfrak{o}$, coincides with $\mathfrak{O}$. If $\mathfrak{a}$ is any subset of $\mathfrak{R}$, $\mathfrak{a}^{-r}[\mathfrak{a}^{-l}]$ is the set of all t in $\mathfrak{R}$ for which $\mathfrak{a}t\subseteqq\mathfrak{O}[t\mathfrak{a}\subseteqq\mathfrak{O}]$; $\mathfrak{a}$ is a left [right] ideal of $\mathfrak{O}$ if (1) $(\mathfrak{a}^{-r})^{-l}=\mathfrak{a}\,[(\mathfrak{a}^{-l})^{-r}=\mathfrak{a}]$ and (2) $\mathfrak{a}^{-r}\cap\mathfrak{o}$ and $\mathfrak{a}\cap\mathfrak{o}$ are not vacuous. If $\mathfrak{a}$ is an ideal of $\mathfrak{O}$, the left-order [right-order] of $\mathfrak{a}$ is the set $\mathfrak{O}_l[\mathfrak{O}_r]$ of all t in $\mathfrak{R}$ for which $t\mathfrak{a}\subseteqq\mathfrak{a}[\mathfrak{a}t\subseteqq\mathfrak{a}]$. If $\mathfrak{a}$ is a left ideal of the maximal order $\mathfrak{O}$, then the left order $\mathfrak{O}_l$ of $\mathfrak{a}$ coincides with $\mathfrak{O}$. The right order $\mathfrak{O}_r$ of $\mathfrak{a}$ is also maximal, and $\mathfrak{a}$ is a right ideal of $\mathfrak{O}_r$; $\mathfrak{a}^{-r}$ is a right ideal of $\mathfrak{O}$ and its left order is the right order of $\mathfrak{a}$. If A is any subset of $\mathfrak{R}$, $\mathfrak{a}_1=(A^{-r})^{-l}$, $\mathfrak{a}_2=(A^{-l})^{-r}$, then $\mathfrak{a}_1$ is a left ideal and $\mathfrak{a}_2$ is a right ideal if conditions (2) hold. If $\mathfrak{a}_1=\mathfrak{a}_2$, then A is said to generate the two-sided ideal $(A)=\mathfrak{a}_1=\mathfrak{a}_2$. If $\mathfrak{a}$ and $\mathfrak{b}$ are two-sided ideals of $\mathfrak{O}$, then the set $\mathfrak{a}\mathfrak{b}$ of all products ab ($a\epsilon\mathfrak{a}$, $b\epsilon\mathfrak{b}$) generates a two-sided ideal $\mathfrak{a}\cdot\mathfrak{b}=(\mathfrak{a}\mathfrak{b})$ and the class sum of $\mathfrak{a}$ and $\mathfrak{b}$ generates a two-sided ideal $(\mathfrak{a}, \mathfrak{b})$. The usual laws for reckoning with ideal sums and products hold, and the two-sided ideals of $\mathfrak{O}$ form a group under the product operation. A definition of proper (eigentlich) product $\mathfrak{a}\cdot\mathfrak{b}$ is given whereby the right-order of $\mathfrak{a}$ coincides with the left-order of $\mathfrak{b}$, and the set of all one-sided ideals of all maximal orders $\mathfrak{O}^*$ satisfying $\lambda\mathfrak{O}^*\subseteqq\mathfrak{O}$, $\mu\mathfrak{O}\subseteqq\mathfrak{O}^*$ (λ, μ in $\mathfrak{o}$) forms a groupoid. If $\mathfrak{R}$ is a group, then the following conditions are equivalent. (1) Every a in $\mathfrak{R}$ generates a two-sided (principal) ideal (a). (2) $a\mathfrak{O}\subseteqq\mathfrak{O}a$ for every a in $\mathfrak{R}$. (3) $\mathfrak{O}$ contains the commutator subgroup $\mathfrak{K}$ of $\mathfrak{R}$. (4) All one-sided ideals of $\mathfrak{O}$ are two-sided. In this case the commutative semigroup $\overline{\mathfrak{O}}=\mathfrak{O}/\mathfrak{K}$ is completely integrally closed in $\overline{\mathfrak{R}}=\mathfrak{R}/\mathfrak{K}$. If $\mathfrak{a}$ is an ideal of $\mathfrak{O}$, then $\bar{\mathfrak{a}}=\mathfrak{a}/\mathfrak{K}$ is a v-ideal of $\overline{\mathfrak{O}}$, and the mapping $\mathfrak{a}\to\bar{\mathfrak{a}}$ is an isomorphism between the ideals of $\mathfrak{O}$ and the v-ideals of $\overline{\mathfrak{O}}$. If $\mathfrak{R}$ is not itself a group, then one considers the group $\mathfrak{Y}$ consisting of all elements a of $\mathfrak{R}$ such that a has an inverse in $\mathfrak{R}$ and generates a principal ideal (a) of $\mathfrak{O}$; $\mathfrak{O}'=\mathfrak{O}\cap\mathfrak{Y}$ is a maximal order in $\mathfrak{Y}$ satisfying the above conditions, and the group of two-sided ideals of $\mathfrak{O}$ is isomorphic with the ideal group of $\mathfrak{O}'$ under the correspondence $\mathfrak{a}\to\mathfrak{a}\cap\mathfrak{Y}$. *A. H. Clifford* (Cambridge, Mass.).

Rybakoff, L. Sur une classe de semi-groupes commutatifs. Rec. Math. (Moscou) **5** (47), 521–536 (1939). (Russian. French summary) [MF 1343]

A commutative semi-group is a system closed under a unique commutative and associative "multiplication" admitting the "cancellation": from $au=av$ it follows that $u=v$. In a pure semi-group no element (the identity, if it exists, excepted) has an inverse. Part of the paper is concerned with the class of semi-groups S which admit a basis: this is a set of elements which generate S, minimal in that property. Within this class the pure semi-groups have a unique basis: the set, namely, of all "prime" elements. More generally, a basis for S consists of a set (g) which generates the group part G of S (consisting of elements which possess an inverse) and a set (h), sharing no element with (g), composed of some of the primes of the semi-group part S' (generated by elements of S not in G). This gives a decomposition of S into factors G and S'' (generated by (h) and the identity). The semi-group S'' is pure. The principal result of the paper is the theorem: a pure commutative semi-group possessing a finite basis and satisfying condition (A) $u^n=v^n$ implies $u=v$ (for every u and v, and integer n) is isomorphic to a divisor of some free semi-group with a finite basis. As to method, it is shown first that an at most finite set of relations can subsist among the given finite set of generators. Next, relations among the generators of the special form $p_i^{a_i}=\Pi_j p_j^{a_j}$, $i\neq j$, are rendered trivial in an enlarged semi-group by the adjunction of new elements q_j such that $q_j^{a_i}=p_j$. Here, by condition (A), it is possible to write $p_i=\Pi_j q_j^{a_j}$. When all possible special relations have been "eliminated," a general one:

$$q_1^{a_1}q_2^{a_2}\cdots q_m^{a_m}=q_{m+1}^{a_{m+1}}\cdots q_n^{a_n},\qquad a_m\geqq\max(a_i),$$

is reduced in complexity by the adjunction of a new element r_1 such that $q_m=r_1q_{m+1}$. This permits a cancellation of the element q_{m+1} from the right side of the relation; after a finite number of steps one reduces the relation to one of special form and applies the first process. The "large" semi-group which is finally obtained contains the original one; no relations which are not trivial subsist among its generators. *L. Zippin* (Flushing, N. Y.).

Eaton, J. E. Theory of cogroups. Duke Math. J. **6**, 101–107 (1940). [MF 1546]

The author abstracts six properties of the special hypergroup (multigroup) whose elements are the left cosets of a finite group relative to a subgroup, and defines a cogroup

as a multiplicative system C for which these properties are postulated. It is shown that a subset H of C which is closed under multiplication is a cogroup; C has a left coset decomposition relative to a subcogroup H; and that the multiplicative system C/H is a cogroup, and is a group if and only if $Hc \supset cH$ for every element c. *H. S. Wall.*

GEOMETRY

★**Amodeo, Federico. Origine e Sviluppo della Geometria Proiettiva.** B. Pellerano, Naples, 1939. 174 pp.

This book follows the history of projective geometry from its origins in the investigations on the theory of perspective drawing, beginning with Piero della Francesca and Dürer, down to the most modern developments. It is divided into two main parts. The first covers the direct development down to the discovery of the non-Euclidean geometries, the stabilization of the idea of complex elements and the use of inversion and general Cremona transformations. In the second, various modern developments are traced: unlimited dimensions, the rigorous axiomatic treatment, continuity, and above all the classification of different kinds of geometry according to the groups with respect to which invariance is considered. The metrical and homographic groups are considered in detail, and in this connexion even such remote geometries as that of the relativity theory, the tensor calculus and Hilbert space are treated.

In each section of the work the author's method is to name as many contributors as possible, with a brief sketch of each man's work; apart from a slight bias in favor of the Neapolitan school, the survey seems to be exhaustive and impartial, as well as being strongly objective and avoiding as far as possible both metaphysical considerations and judgments of value. Full bibliographical references are given in footnotes, and there is an index of names which makes the work convenient for reference. It is unfortunate, however, that almost every page offers at least one misprint.

P. Du Val (Manchester).

Sanguineti, Jerónimo. Contribution to the study of some special curves. An. Soc. Ci. Argentina **128**, 71–81 (1939). (Spanish) [MF 1082]

Some elementary properties of a quartic C with the polar equation $\rho^2 = 1/\cos\omega$. Drawing only one of the two symmetric branches of C, the author supposes similitude to the versiera, the equation of which is erroneously given. Two curves derived from C are likewise incomplete; their generation could easily be reduced to the cissoid construction (adding of radii) from C and one or two circles.

A. E. Mayer (London).

Bottema, O. Eine Geometrie mit unvollständiger Anordnung. Math. Ann. **117**, 17–26 (1939). [MF 1321]

Let the foundations of plane affine geometry be only so far prescribed as to insure that attached to each point is an ordered pair of elements x, y of a field K with characteristic p different from 2. The equation $a_{11}x^2+2a_{12}xy+a_{22}y^2+2a_{13}x+2a_{23}y+a_{33}=0$ with non-vanishing determinant is an ellipse E provided $a_{11}a_{22}-a_{12}^2$ is a non-zero square, and a point P is inside E provided each line through P has two distinct points in common with E. If two distinct tangents of E go through P, then the point is outside E. Defining the center of E in the usual way, the author introduces the axiom: (I). The center of an ellipse is inside the ellipse. The paper is devoted to establishing some interesting consequences of this axiom which induces in the geometry an "incomplete" order possessing many of the important properties of "complete" order as given, say, by Hilbert's second group of axioms. As an easy consequence of (I), the field K has the property that the sum of two squares, not both zero, is a non-zero square. This eliminates finite fields as well as the field of rational numbers and the field of complex numbers. The reader is referred to the paper for a list of the many familiar consequences of the usual Axiome der Anordnung which retain their validity when merely the "incomplete" order given by (I) is assumed. Among the order relations which are not valid in this weaker system may be mentioned (a) if $B \neq D$ are two points of an interval (A, C), then D is either in the interval (A, B) or in the interval (B, C), and (b) if C_1 and C_2 are two circles without a common point, then either C_1 is outside C_2 or C_2 is outside C_1.

L. M. Blumenthal (Columbia, Mo.).

Gorn, Saul. On incidence geometry. Bull. Amer. Math. Soc. **46**, 158–167 (1940). [MF 1264]

The author shows how to introduce ideal elements (points, lines, planes, $\cdots$ "at infinity") in a general "incidence geometry," the cases of affine and descriptive geometry being included. His fundamental assumption is that the elements above a point form a projective geometry (centered affine geometry). His ideal elements other than hyperplanes are "maximal equi-transversal classes." In the case of three dimensions, he needs to assume also that if there is a line skew to a whole pencil of lines, then there is a plane through this line cutting every member of the pencil, but not their common point. The arguments are lattice-theoretic. *G. Birkhoff* (Princeton, N. J.).

Rickart, C. E. The Pascal configuration in a finite projective plane. Amer. Math. Monthly **47**, 89–96 (1940). [MF 1390]

Six points on a conic, taken in the sixty possible cyclic orders, form sixty hexagons. The point of intersection of two opposite sides of such a hexagon is called a Pascal point, and the line on which the three Pascal points lie is called a Pascal line. If it happens that three alternate sides of the hexagon concur, the point of concurrence counts for three among the 45 Pascal points, and is called a triple point. For instance, the configuration determined by a regular hexagon inscribed in a circle has one triple point at the center, and three others at infinity. It is impossible for a Pascal configuration in the real plane to have more than four triple points. The author proves that the same result holds in the modular plane $PG(2, p)$, where $p \equiv 3 \pmod 4$. (Since a conic in the modular plane has only $p+1$ points, we must stipulate $p \geqq 5$.) For other values of p, greater than 5, the maximum number of triple points is shown to be six. But every Pascal configuration in $PG(2, 5)$ has exactly ten triple points, and these with their polars form a Desargues configuration. In this case the sixty Pascal lines coincide by sixes on the polars of the triple points.

H. S. M. Coxeter (Toronto, Ont.).

Mittelstaedt. Affine Vierecksinvarianten, dreieckssymmetrisch (zyklisch) betrachtet. Allg. Vermessgs.-Nachr. **51**, 648–654 (1939). [MF 1399]

With every quadrilateral $AMIN$ a triangle AA_1A_2 is

connected, if A_1 and A_2 are the points of intersection of the two pairs of opposite sides AN, MI and AM, NI. The author shows in this paper that many of the functions of the quadrilateral, which are important in surveying, can be expressed, in a very simple way, by three quantities h_1, h_2, h_3 defined by the situation of the point I with respect to AA_1A_2. The author seems not to be aware that the quantities h_1, h_2, h_3 are identical with the Moebius' barycentric coordinates of the point I with respect to the triangle AA_1A_2. A nomogram is added to give the connections between different functions of the quadrilateral.

E. Helly (Paterson, N. J.).

Robinson, R. T. Theorems on perspectivity. Math. Gaz. **24**, 9–14 (1940). [MF 1456]

Using projective coordinates with $ABCD$ as tetrahedron of reference, the author proves the theorem: Any plane cuts the edges of a tetrahedron $ABCD$ in six points, forming a complete quadrilateral. If UVW is the triangle formed by the diagonals of this quadrilateral, then (1) the triangular faces of the tetrahedron are each of them in perspective with the triangle UVW from points A_1, B_1, C_1, D_1, respectively; (2) the tetrahedra $ABCD$, $A_1B_1C_1D_1$ are in perspective from U, V, W and a fourth point Ω. These statements are used to derive a few related theorems.

E. Helly (Paterson, N. J.).

Weitzenböck, R. Ein Satz über assoziierte Geraden im R_4. Nederl. Akad. Wetensch., Proc. **43**, 13–17 (1940). [MF 1234]

If four general straight lines in a R_4 are given, a fifth line can be constructed linearly to obtain the known figure of five associated lines in the R_4. Using the methods of the theory of the projective invariants the author proves: (1) Five general associated lines cannot be generators of the same F_2 in the R_4. (2) The fifth line is the locus of all three-dimensional quadric cones, which contain the first four lines.

E. Helly (Paterson, N. J.).

Miyazaki, Sadataka. Doppelberührungslehre der Kurven zweiter Ordnung. I. Jap. J. Math. **16**, 135–147 (1939). [MF 1107]

The author refers to two previous papers of his on this topic [Tôhoku Math. J. (1936); Proc. Phys.-Math. Soc. Japan (1938)]. Double contact between two curves of second order K_1, K_2 is defined by the existence of a singular line-pair (repeated line) linearly dependent on K_1, K_2 and distinct from both of them; they are said to touch doubly on this line. If K touches K_1, K_2 doubly on lines L_1, L_2, there is a relation

$$\alpha K_1-\beta K_2=L_1^2-L_2^2,$$

and the GDK (common double contact curve) K of K_1, K_2 is said to belong to the line pair $\alpha K_1-\beta K_2$; GDK's belonging to the same line pair are called similar, and form a GDM (similar double contact manifold) of K_1, K_2. If K, K_1 are similar GDK's of S_2, S_3, K, K_2 of S_3, S_1, and K, K_3 of S_1, S_2, then S_1, S_2, S_3 are called perspective to K_1, K_2, K_3, and K is called the center of perspective. The Grundsatz is: If S_1, S_2, S_3 are perspective to K_1, K_2, K_3, then K_1, K_2, K_3 either (1) are perspective to S_1, S_2, S_3, or (2) all pass through two fixed points (in special cases through one), or (3) are line-pairs all containing a common line, or (4) one of them has double contact with the two others, or (5) all are singular line-pairs.

The other principal results (out of 21 numbered Sätze, almost all trivial) are: If every two out of four curves of the second order are similar GDK's of the remaining two, then all four belong to a pencil, unless they are all line-pairs. If every two of S_1, S_2, S_3 are similar GDK's of every two of K_1, K_2, K_3, all six belong to a pencil.

The familiar representation of conics by points of 5-dimensional space is sketched; it is pointed out that a GDM is then represented by a conic, and the number of GDM's of two curves in different cases is obtained. *P. Du Val.*

Algebraic Geometry

Kommerell, Karl. Beweis eines Fundamentalsatzes der Kurven 3. Ordnung. Math. Z. **45**, 756–758 (1939). [MF 1413]

This well-known theorem is proved: A pencil of cubics is defined by eight points, no seven of which lie on a conic and no four on a line, and all curves of the pencil have a unique ninth point in common. The new feature of the proof consists in showing that the following assumption is untenable: All cubics through seven of the eight points, chosen as in the above theorem, must pass through the eighth point. *T. R. Hollcroft* (Aurora, N. Y.).

Châtelet, François. Groupe exceptionnel d'une classe de cubiques. C. R. Acad. Sci. Paris **210**, 200–202 (1940). [MF 1632]

Using definitions, methods and results of earlier papers [C. R. Acad. Sci. Paris **206**, 1532–1533 (1938); **209**, 90–92 (1940)], the author constructs sets of groups of exceptional points of classes of cubic curves in an algebraic field k', when a curve in a field k is given. Several illustrations of the process are given. *V. Snyder* (Ithaca, N. Y.).

Châtelet, François. Points exceptionnels d'une cubique de Weierstrass. C. R. Acad. Sci. Paris **210**, 90–92 (1940). [MF 1239]

The theorem that the rational points of an elliptic cubic curve with rational coefficients are integers was generalized by Miss Lutz [J. Reine Angew. Math. **177**, 238–247 (1937)] to the case in which the coefficients are integers in a p-adic algebraic field K. They form an infinite abelian group G. The present paper applies similar methods to determine the coordinates of the points of each finite sub-group of G (exceptional points). *V. Snyder* (Ithaca, N. Y.).

Fano, Gino. Quelques remarques à propos d'une Note de M. Amin Yasin. C. R. Acad. Sci. Paris **210**, 284–285 (1940). [MF 1598]

A plane non-singular quintic curve has 2015 contact conics. Of these, 991 were found by F. P. White [Proc. London Math. Soc. (2) **30**, 347–358 (1930)] by means of cubic primals in [3], and by Amin Yasin [C. R. Acad. Sci. Paris **209**, 337–338 (1939)] who employed quartic surfaces in [3]. The author shows that these proofs are in essence identical. A cubic primal in [4] has 495 prime sections having four nodes. The quartic circumscribing cones of these sections from a node are composite, consisting of two quadric cones. These and other contact conics are included in a system of contact quartics. *V. Snyder.*

Kubota, Tadahiko. Characteristic properties of algebraic figures. Tensor **2**, 1–7 (1939). (Japanese) [MF 1362]

A summary report of some geometrical properties which are characteristic for algebraic curves.

A. Kawaguchi (Sapporo).

Amodeo, Federico. Nuovo metodo per la geometria delle serie lineari delle curve algebriche. Rend. Sem. Mat. Fis. Milano (4ª) 9, 1–24 (1939). [MF 1295]

After 38 years, the author returns to the study of linear series on algebraic curves in order to reëstablish the theory on a purely geometric basis, in accordance with the following statement: "Above all, it is shown that the theory of linear series, born of Abelian integrals and Riemann surfaces, has almost no longer any need of this advanced analytic theory and can be obtained directly from the 'Teoria generale delle curve piane' of the great Cremona." This paper deals only with curve systems whose double points are independent in position. The author states that a later paper will consider curves some of whose double points have positions depending upon those of others.

T. R. Hollcroft (Aurora, N. Y.).

Farina, Mariantonia. Sulle curve piane, algebriche, reali che presentano "massimi d'inclusione." Ist. Lombardo, Rend. 72, 85–90 (1939). [MF 944]

Verfasser zeigt die Existenz reeller ebener algebraischer Kurven $(4m+3)$ter Ordnung, welche die maximale Anzahl reeller Züge besitzen und auch das Maximum $(m+1, m)$ der Einschliessung zeigen. Die von Verfasser angegebenen ebenen Kurven $(4m+3)$ter Ordnung haben nämlich die maximale Anzahl reeller Züge und sie besitzen überdies die folgende Eigenschaft. Es gibt einen geraden Zug α mit einem inneren Punkte P und einen geraden Zug β (sogar auf zwei Arten) mit einem inneren Punkte Q, welche keine andere Züge einschliessen, derart dass es $m+1$ P und nicht Q einschliessende Züge α und m Q und nicht P einschliessende Züge β gibt. Die Reihe der Züge α kann hier derart geordnet werden, dass jeder Zug den folgenden einschliesst; dasselbe gilt für die zwei Reihen der Züge β. Diese drei Reihen liegen ausserhalb einander. Dieses Resultat ergänzt ein Ergebnis von G. Biggiogero [Ist. Lombardo, Rend. (2) 55, 499–510 (1922) und 56, 841–849 (1923)].

G. Schaake (Groningen).

Bompiani, E. Teoremi sulle Jacobiane di particolari reti di curve piane e loro estensioni a particolari sistemi lineari di forme. Ist. Lombardo, Rend. 72, 362–370 (1939). [MF 932]

Die Jacobische Kurve J_2 eines Netzes von ebenen Kurven $n$ter Ordnung welche einander in n verschiedenen kollinearen Punkten von $(n-2)$ter Ordnung berühren und überdies durch einen nicht auf der Geraden l der genannten n Punkte liegenden Punkt B der Ebene gehen, ist eine Kurve $n$ter Ordnung, welche durch die n genannten Punkte von l geht; die Tangente in einem dieser Punkte wird bestimmt durch das Doppelverhältniss $(n-2)/(n-1)$, das sie bildet mit der Tangente der Netzkurven, der Verbindungsgeraden mit B und der Geraden l. In B hat J_2 dieselben nodalen Tangenten wie die Netzkurve, welche dort einen Doppelpunkt besitzt, und ihre Kontaktinvariante (Verhältniss der Krümmungen) mit jedem nodalen Zweige dieser Netzkurve ist 2. Die J_2 kann aus einer beliebigen Netzkurve abgeleitet werden durch Anwendung einer quadratischen Transformation mit zwei auf l liegenden Fundamentalpunkten und einem Fundamentalpunkte in B; die Polare von B in Bezug auf die Transformierte der Netzkurve geht durch die Inverse der quadratischen Transformation in J_2 über. Verfasser betrachtet auch das System der Hyperflächen $n$ter Ordnung in einem S_k, die mit einem S_{k-1} denselben Durchschnitt haben, in den Punkten dieses Durchschnitts einander von der Ordnung r berühren und durch einen festen Punkt B gehen; J_{n-r} ist dann der Ort der $(n-r)$-fachen Punkte dieser Hyperflächen. Verfasser leitet die Gleichung von J_{n-r} ab; J_{n-r} und eine Hyperfläche des Systems schneiden S_{k-1} in denselben Punkten einer V_{k-2}. Die Berührungshyperebene von J_{n-r} in einem Punkte P von V_{k-2}, die Berührungshyperebene einer Hyperfläche des Systems in P, die Hyperebene, welche den Berührungs-S_{k-2} von V_{k-2} in P mit B verbindet und S_{k-1} haben das Doppelverhältniss $r/(n-1)$.

In B haben J_{n-r} und die Systemhyperfläche, welche in B einen $(n-r)$-fachen Punkt hat, denselben Tangentialkegel und zwei einander in B berührende Kurven beider Hyperflächen, welche in B dieselbe Oskulationshyperebene besitzen, haben dort die Kontaktinvariante $n-r$. Durch Anwendung einer Koordinateninversion mit k Fundamentalpunkten in S_{k-1} und einem Fundamentalpunkte in B auf eine Hyperfläche des Systems bekommt man eine Hyperfläche; die $(n-r-1)^t$ Polare von B in Bezug auf die letztgenannte Hyperfläche gibt, wenn man darauf die inverse Transformation anwendet, J_{n-r}.

G. Schaake.

Brusotti, Luigi. Fasci reali di curve algebriche sopra una quadrica reale. Ist. Lombardo, Rend. 72, 3–9 (1939). [MF 940]

The following theorem is proved: In every family of algebraic curves on a real quadric with hyperbolic points, there are pencils of curves, algebraically generated, all of whose base points and critical centers are real. This theorem does not hold for quadrics with elliptic points.

T. R. Hollcroft (Aurora, N. Y.).

Shreve, Darrell R. On a certain class of symmetric hypersurfaces. Bull. Amer. Math. Soc. 45, 948–951 (1939). [MF 789]

The author considers the primals of S_r invariant under the symmetric permutation group on $r+2$ superabounding coordinates $(x_1, x_2, \cdots, x_{r+2})$ which satisfy identically the relation $\sum_{i=1}^{r+2} x_i = 0$, and particularly the primals of odd order $n \geqq 3$ represented by $\sum_{i=1}^{r+2} x_i^n = 0$; he gives, without demonstrations, some general properties concerning their double points, as well as the linear spaces and the Eckardt points belonging to them (simple points, the tangent prime of which is contained as a part in the first polar).

B. Segre (Cambridge, England).

Szőkefalvi Nagy, Gyula. Über das Geschlecht der einschaligen Flächen vom Maximalindex. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 58, 298–312 (1939). (Hungarian. German summary) [MF 1018]

In previous papers the author proved the existence of surfaces with one sheet and of maximum index whose genus is 0 or 1. In the present paper surfaces of order n are constructed whose genus is $n-2$ or $n-3$.

G. Szegö.

Bonera, Piero. Sui punti doppi impropri delle superficie razionali nello spazio a quattro dimensioni. Ist. Lombardo, Rend. 72, 73–84 (1939). [MF 943]

The author considers the rational surfaces F, in [4], representable upon a plane by means of some ∞^4 linear system Σ of curves satisfying certain restrictions; he expresses the number d of the improper double points of F as

a function of the characters of Σ, and, making use of simple arithmetical arguments, determines 8 types of surfaces F having $d=0$, without, however, excluding the existence of other types. *B. Segre* (Cambridge, England).

Villa, Mario. Ricerche sulle varietà V_k che posseggono $\infty^\delta E_2$ di $\gamma_{1,s}$, con particolare riguardo al caso $k=4$, $\delta=8$. Mem. R. Accad. Naz. Lincei 7, 373–427 (1939). [MF 515]

An "E_2 of $\gamma_{1,s}$" belonging to a V_k is an element E_2 of the second order (point, tangent and osculating plane) attached to a curve on V_k whose osculating S_s at the point is either indeterminate or lies in an S_{k+1} containing the tangent S_k to V_k at the point. The author determines the V_4 which contain ∞^8 such elements; such a variety is either (i) a V_4 of S_8, lying on the cone projecting a Veronese surface from a plane not meeting its ambient S_5, and meeting the generating S_3 of the cone in surfaces; (ii) the V_4^6 of S_8 which represents the pairs of points of two planes; (iii) any V_4 of S_7 whose tangent S_4's fill the whole S_7; (iv) a cone projecting a V_3 containing $\infty^4\gamma_{1,s}$'s from a point outside its ambient space. The latter part of the paper discusses certain varieties of $k>4$ dimensions containing $\infty^\delta E_2$ of $\gamma_{1,s}$, where $\delta>2k-1$. In all the cases examined the E_2 can be derived as the osculating elements of a set of quasi-asymptotic curves $\gamma_{1,s}$ depending on arbitrary functions.

J. A. Todd (Cambridge, England).

Villa, Mario. Nuove ricerche nella teoria delle curve quasi-asintotiche. Ann. Mat. Pura Appl. 18, 275–308 (1939). [MF 923]

In extension of investigations carried out in the above paper the author determines the maximum ν and the minimum μ of linearly independent partial differential equations of order $s-1$ satisfied by a V_k possessing $\infty^\delta E_2$ of $\gamma_{r,s}$ $(2k-1\leqq\delta<3k-2;\ 0<r<s-1)$, where k, δ, r, s are given, and studies the varieties which satisfy exactly ν or μ such equations. The majority of the author's results are too involved to cite in a review; we mention one typical theorem. The only V_k's possessing $\infty^\delta E_2$ of $\gamma_{s-2,s}$, whose ambient space has the greatest dimension consistent with this hypothesis, are the varieties in space of $\rho_0+3k-\delta-1$ dimensions whose osculating spaces $S(s-2)$ have dimension ρ_0 and completely fill the ambient space. The author shows that

$$\mu=\binom{k+s-2}{s-1}-\binom{3k-\delta+s-3}{s-1},$$

and that

$$\nu=\binom{k+s-2}{s-1}-(3k-\delta-1),$$

and determines explicitly the corresponding varieties.

J. A. Todd (Cambridge, England).

Finsler, Paul. Über die Darstellung und Anzahl der Freisysteme und Freigebilde. Monatsh. Math. Phys. 48, 433–447 (1939). [MF 660]

An algebraic variety V is called by the author a free-variety (Freigebilde) if there does not exist any linear space intersecting it in a finite number of linearly dependent points (no intersection being counted more than once); if, in particular, V is composed of a finite number of linear spaces, then it is called a free-system (Freisysteme). The author met with these notions in studying the question (connected with Legendre's conditions for n-dimensional variation problems) of when a given linear system of quadratic forms contains a definite form [Comment. Math. Helv. 9, 188 (1937)]; and he has already devoted some other papers to them [Comment. Math. Helv. 9, 172 (1937) and 11, 62 (1938)]. Here he finds that there are, respectively, 1, 1, 2, 5, 16, 66, 354 different types of free-systems belonging to a space of dimension 0, 1, 2, 3, 4, 5, 6 (and not to one of lower dimension); these types are classified and stated in detail at the end, by means of a convenient notation, together with the different types of connected 1-dimensional free-varieties belonging to such spaces (of which there are 0, 1, 2, 4, 10, 26, 79, respectively). He shows moreover how to find in these spaces 1, 2, 5, 15, 51, 226, 1266 types of (connected or not connected) free-varieties of the different dimensions, without however prejudicing the question of the possible existence of other types. *B. Segre.*

Lense, Josef. Beiträge zur Theorie der isotropen Mannigfaltigkeiten. Monatsh. Math. Phys. 48, 121–128 (1939). [MF 631]

Zuerst beweist Verfasser den Satz: Ist V_m eine in einer euklidischen R_n gelegene isotrope Mannigfaltigkeit vom Range Null mit ∞^ρ Tangentialräume und ist $\rho>n-2m$, so liegt die V_m in einer isotropen R_s vom Range $q<s-(m-\rho)$, wobei $m\leqq s<m+\rho$. Es werden weiter isotrope V_m vom Range r betrachtet, welche das Erzeugnis der Schmieg-R_{m-1} einer Kurve $y^h=y^h(u)$ sind. Es sei G_μ die Determinante

$$\sum_h y_{(\alpha)}{}^h y_{(\beta)}{}^h, \qquad \alpha,\beta=1,\cdots,\mu;\quad y_{(\alpha)}{}^h=\frac{d^\alpha y^h}{du^\alpha}.$$

Es ist dann r der Rang der Determinante G_m. Insbesondere wird der Fall betrachtet, wobei $r=0$ ist und die Determinanten $G_1,\cdots,G_{2m}$ verschwinden. *J. Haantjes.*

Todd, J. A. The postulation of a multiple variety. Proc. Cambridge Philos. Soc. 36, 27–33 (1940). [MF 896]

The postulation of a multiple curve of order n and genus p on a primal in S_r is obtained by considering a degenerate form of the curve consisting of n lines, of virtual genus p and with $d=n+p-1$ intersections. A similar method is used to determine the postulation of a multiple surface on a primal in S_4, confirming the result previously obtained by Roth. The method may be extended to finding the postulation of varieties of higher dimension on primals, but the formulas rapidly become unwieldy. *T. R. Hollcroft.*

Differential Geometry

Stoker, J. J. Unbounded convex point sets. Amer. J. Math. 62, 165–179 (1940). [MF 966]

On considère dans l'espace E^3 les ensembles S non bornés convexes fermés, ayant des points intérieurs et ne comprenant pas tout E^3. Les demi-droites passant par un point donné de S, et dont tous les points appartiennent à S, forment un cône convexe; les cônes formés à partir de deux points différents se déduisent l'un de l'autre par translation. Un ensemble S est (1) ou bien l'ensemble des points compris entre deux plans parallèles; (2) ou bien homéomorphe à un cylindre de révolution; (3) ou bien homéomorphe à un demi-espace. L'image sphérique de S est dans un hemisphere fermé. Si S est du type (3), on peut trouver un plan (x, y) tel que la frontière de S puisse être définie en coordonnées rectangulaires par $z=f(x, y)$, avec f univoque, continue (et non-concave). *J. Favard* (Grenoble).

Gericke, H. Über stützbare Flächen und ihre Entwicklung nach Kugelfunktionen. Math. Z. 46, 55–61 (1940). [MF 1477]

Let K be the Gauss curvature of a closed convex surface F, O the center of gravity determined by the mass distribution with the density function K. Let T be an axis through O for which the moment of inertia of this distribution is stationary. By use of the Laplace expansion of the "supporting function" all the surfaces are determined for which the axis T keeps the property mentioned when F is replaced by an arbitrary parallel surface. *G. Szegö.*

Haupt, Otto. Geometrische Ordnungen. Jber. Deutsch. Math. Verein. 49, 190–207 (1940). [MF 1214]

Unter der Ordnung einer Punktmenge $\mathfrak{M}$ versteht man die maximale Mächtigkeit der Durchschnitte von $\mathfrak{M}$ mit einer Schar ausgezeichneter Punktmengen, der sogenannten Ordnungscharakteristiken. In den klassischen Untersuchungen C. Juels und in zahlreichen anschliessenden werden für $\mathfrak{M}$ Bögen, Flächen, etc. in einem projektiven Raume gewählt, die mehr oder weniger starken Differenzierbarkeits- und Regularitätsbedingungen unterworfen werden. Der Autor berichtet über ausgewählte neuere Fragestellungen und Untersuchungen hauptsächlich des Erlanger Kreises. Ihre gemeinsame Tendenz ist, die Voraussetzungen über $\mathfrak{M}$ zu verringern und den Ordnungsbegriff zu verallgemeinern. Hervorgehoben seien zwei den Anschluss an die Topologie vermittelnde intrinseque Ordnungsdefinitionen: An den Ordnungscharakteristiken wesentlich ist nur die von ihnen auf $\mathfrak{M}$ induzierte Involution. Linsman definiert die Ordnung von $\mathfrak{M}$ und von Teilmengen von $\mathfrak{M}$ in abstracto durch Vorgabe einer Involution auf $\mathfrak{M}$. Noch allgemeiner setzt Haupt $\mathfrak{M}$ als topologischen Raum voraus, in dem ein alle Umgebungen umfassendes System von Teilmengen gegeben und jeder Teilmenge eine Ordnungszahl zugeordnet ist, die bei wachsender Teilmenge nicht abnimmt. Es werden zahlreiche Fragen des Verhaltens im Kleinen ("Struktur") und Grossen ("Gestalt") besprochen, ferner Beziehungen zur Differential- und algebraischen Geometrie und zur Theorie der reellen Funktionen. In der Bibliographie fehlt der Name B. Segres. *P. Scherk* (New York, N. Y.).

Marchaud, A. Sur les surfaces du troisième ordre de la géométrie finie. J. Math. Pures Appl. 18, 323–362 (1939). [MF 1274]

L'ordre d'un ensemble est le nombre maximal des points qu'il peut avoir en commun avec une droite qui n'appartient pas entièrement à l'ensemble. L'auteur étudie des ensembles fermés S du troisième ordre dans l'espace projectif réel à trois dimensions, toutes les sections planes duquel se composent de courbes de Jordan fermées avec ou sans points doubles et d'un point isolé [compté pour deux] au plus; du moins une de ces sections doit contenir une courbe de Jordan d'ordre trois. Un point de S est dit régulier s'il y a une droite par ce point rencontrant S exactement en trois points, autrement irrégulier. Une droite dont tous les points sont irrégulier est appelée irrégulière. L'auteur montre que le voisinage de tout point régulier de S est un morceau de surface simple de Jordan remplissant une condition de Lipschitz. L'existence d'un plan tangent dans le point est discutée. Quant aux points irréguliers, abstraction faite d'un point isolé et du cas que S contient un élément cônique (alors S contient un cône entier), seulement deux cas sont possibles: (1) Il n'y a pas de droite irrégulière; alors il y a quatre points irréguliers au plus et S n'est pas réglée. (2) S possède une droite irrégulière Δ et deux points irréguliers en dehors d'elle au plus. Alors S est une surface réglée engendrée par la variation continue d'une droite rencontrant Δ; S a partout un plan tangent continue sauf le long de Δ et de deux génératrices au plus. *P. Scherk* (New York, N. Y.).

Haupt, Otto, Nöbeling, Georg und Pauc, Christian. Über Abhängigkeitsräume. J. Reine Angew. Math. 181, 193–217 (1940). [MF 1210]

The authors first describe their theory as a generalization of Bouligand's theory of paratangents. They then found an abstract theory of "dependence," showing its fruitfulness by many examples. The first part of the discussion is along conventional lines [cf., for example, H. Whitney: An abstract theory of linear dependence, Amer. J. Math. 57, 509–533 (1935); also T. Nakasawa: Zur Axiomatik der linearen Abhängigkeit, Sci. Rep. Tokyo Bunrika Daigaku 2, 235–255 (1935) and 3, 45–69 and 123–136 (1936)]; there is a discussion of bases, rank, etc. The ideas are correlated by the authors with lattice theory [cf. also G. Birkhoff: Abstract linear dependence and lattices, Amer. J. Math. 57, 800–804 (1935)]. The paper concludes with interesting equivalent formulations of the property of direct indecomposability, including an ingenious sufficient (but not necessary) condition. *G. Birkhoff* (Princeton, N. J.).

Kasner, Edward and De Cicco, John. General trihornometry of second order. Proc. Nat. Acad. Sci. U. S. A. 25, 479–481 (1939). [MF 880]

Three plane curves having a common point, tangent and circle of curvature constitute a trihorn of the second order, and determine three dihorns of the second order. Each such dihorn has three conformal invariants, a "distance," a direct and a reverse dihorn angle. Regarding the nine invariants of a trihorn as analogous to the parts of a triangle, the authors state nine theorems analogous to those of plane trigonometry. Four of these theorems serve to express any fourth dihorn angle in terms of any given three. One relation is analogous to the law of sines, and the other four are analogous to the law of cosines. The theorems show that any set of four "parts" of a trihorn (of which at most three are dihorn angles) determine the remaining five parts. *P. Franklin* (Cambridge, Mass.).

De Cicco, John. An analog of the nine-point circle in the Kasner plane. Amer. Math. Monthly 46, 627–634 (1939). [MF 976]

A simple horn-set consists of all curves which possess a common point and a common direction. Any curve of such set is defined by a pair of numbers (x, y), where x is curvature and $y=dx/ds$, ds being the arc length. This pair (x, y) denotes a point in the Kasner plane. Conformal transformations induce an affine G_3 in the Kasner plane. The distance of two points in this plane is the conformal measure of the horn angle of the two corresponding curves. This brings into the Kasner plane an isotropy, and there exist parabolic circles. This paper gives theorems about the circumscribed, the inscribed and the six-point parabolic circles of a general triangle. The six-point parabolic circle is an analog of the nine point circle in ordinary geometry; it passes through the three feet of the medians and the three feet of the zero lines through the vertices of the triangle. This Kasner geometry will be more fully developed in a forthcoming book by Kasner and the author. [See also

Kasner, Science **85**, 480–482 (1937) and Proc. Nat. Acad. Sci. U. S. A. **23**, 337–341 (1937).] *D. J. Struik.*

De Cicco, John. The geometry of fields of lineal elements. Trans. Amer. Math. Soc. **47**, 207–229 (1940). [MF 1583]

In a series of papers (cited in the present one) Kasner and De Cicco have developed the differential geometry of oriented lineal elements in the plane under the group G_6, a combination of the whirl group W_3 and the rigid plane motions M_3. In this paper De Cicco obtains the analogues of many results of ordinary surface theory. While the geometric meaning is novel, the results in form are identical with the classical ones. *P. Franklin.*

Rangachariar, V. On the three conicoids of a tangential system which pass through a point. Math. Student **7**, 97–100 (1939). [MF 1563]

Bompiani, Enrico. Geometria differenziale e geometria algebrica. Atti Accad. Peloritana **41**, 117–120 (1939). [MF 1330]

Various extensions of Bäcklund's theorem concerning a secant line of a plane curve [Jber. Deutsch. Math. Verein. **49**, 143–145 (1939)] are given, emphasizing the projective nature of the theorem, in terms of projective invariants of pairs of elements belonging to the plane. The validity of results obtained by the methods of projective differential geometry in the study of algebraic entities is featured. *V. Snyder* (Ithaca, N. Y.).

Abramescu, Nicolas. Nouvelle méthode pour obtenir la cubique qui donne les tangentes de Darboux en un point d'une surface. C. R. Acad. Sci. Paris **209**, 780–781 (1939). [MF 863]

Let C be a curve of normal section of a surface S through a point M on S. Let M_1, M_2 be points at distances of s and $-s$ from M measured along C. The line M_1M_2 intersects the tangent to C at M in a point T_s. The limit of T_s as S tends to zero is a point T distinct from M. The locus of T as C varies through the set of normal sections is a rational cubic curve. The lines joining M to the points of inflexion of this cubic curve are the tangents of Darboux. *V. G. Grove* (East Lansing, Mich.).

Bompiani, Enrico. Le superficie d'ordine $r-1$ dello spazio ad r dimensioni. Boll. Un. Mat. Ital. (2) **2**, 10–18 (1939). [MF 1306]

By differential methods the author gives a new proof of the theorem of Del Pezzo that surfaces of order $r-1$ in $[r]$ are normal ruled surfaces, or, for $r=5$, may be surfaces of Veronese [Rend. Accad. Sci. Fis. Mat. Napoli (1) **24**, 212–216 (1885)]. The minimum directrix curves of the surfaces are found, and a minimum analytic representation of the images of their prime sections is given. *V. Snyder.*

Takeda, Kusuo. On line congruences, III. Tôhoku Math. J. **46**, 46–67 (1939). [MF 1167]

The first paper of the series was published in the Tôhoku Math. J. **44**, 356–359 (1938), the second in **45**, 103–110 (1939). The theory in these papers is based on a completely integrable system of differential equations satisfied by the Plücker coordinates of the lines of the congruence. These coordinates, as is customary, are also interpreted as the coordinates of a point on a hyperquadric Q_4 in a projective space S_5. The congruence K is studied partially by means of its image V_2 on Q_4. Two congruences K and K_1 whose lines correspond in such a manner that the two families of developables of each correspond to the two families of developables of the other are said to correspond asymptotically. A relationship is found between an asymptotic correspondence and a projective deformation of pairs of congruences studied by S. Finikoff [C. R. Acad. Sci. Paris **199**, 177 (1934)]. Quadratic complexes having contact of various orders with the given congruence are derived. In particular, one pair, denoted by $\mathfrak{C}_2^3$, $\mathfrak{C}_2^4$, having second order contact and completely characterized by other geometrical properties, is derived. Associated with each of these latter complexes is a form which has been given a geometrical interpretation. If one or the other of these forms vanishes identically, one or the other of the focal surfaces is a quadric, and conversely. *V. G. Grove.*

Behari, Ram. On the generators of a ruled surface. Tôhoku Math. J. **46**, 41–43 (1939). [MF 1165]

Einfacher Beweis (mit den üblichen Formeln für Regelflächen) einer (bekannten) Differentialbeziehung zwischen der Totalkrümmung einer Regelfläche und der geodätischen Krümmung der Orthogonaltrajektorien der Erzeugenden. Der Satz, dass auf der gemeinen Schraubenfläche die Totalkrümmung längs der Orthogonaltrajektorien der Erzeugenden konstant ist, wird auf alle Regelflächen mit einer gewissen Eigenschaft der mittleren Krümmung ausgedehnt (durch Umformung einer Mainardi-Codazzi-Gleichung). *H. Samelson* (Zürich).

Calapso, Riccardo. Sulle superficie sviluppabili. Atti Accad. Peloritana **41**, 27–31 (1939). [MF 1324]

Let γ be a curve in a space S of $n \geqq 3$ dimensions; the problem is to determine the developable hypersurfaces which contain the curve γ. This problem is very simple; and Guichard [Les courbes de l'espace à n dimensions, Mémor. Sci. Math., fasc. 29, 1928] and Bompiani [Sulle curve sghembe; Scritti matematici offerti al Prof. Berzolari, Pavia, 1936] have found two elementary methods for obtaining solutions of the stated problem. In this paper the author demonstrates that the two methods are closely related to each other and lead to the same formulae. *G. Fubini* (Princeton, N. J.).

Lane, Ernest P. A theorem on surfaces. Bull. Amer. Math. Soc. **46**, 117–120 (1940). [MF 1257]

Let the curves of the two families of asymptotic curves on an analytic non-ruled surface S be called the u-curves and the v-curves. Let the ruled surfaces composed of the tangents to the u-curves (v-curves) at the fixed points of a v-curve (u-curve) be denoted by R_u (R_v). The author proves the following theorem: If the u-curves and v-curves on S belong to linear complexes, then the u-curves (v-curves) are projectively equivalent not only to each other, but also to all of the non-rectilinear asymptotic curves on all of the ruled surfaces R_v (R_u). As corollaries it follows that the u-curves on S belong to linear complexes if, and only if, the asymptotic curves on R_v also belong to linear complexes. Moreover, the u-curves on S are twisted cubics if, and only if, the asymptotic curves on R_v are twisted cubics.

In proving the theorem, the author shows, by using an ingeniously chosen parametric representation for the surfaces, that the fourth order differential equation characterizing the u-curve (v-curve) on the given surface is independent of the parameter for the v-curve (u-curve).

The same differential equation is satisfied by the projective coordinates of an arbitrary point on the non-rectilinear asymptotic curves on R_v in a properly chosen parametric representation for those curves on that surface.
V. G. Grove (East Lansing, Mich.).

Bol, G. Solution of prize-problem 13, 1934. Nieuw Arch. Wiskde **20**, 113–162 (1940). (Dutch) [MF 1092]

The prize problem 13, 1934 of the Amsterdam Mathematical Society first defined a plane net of curves as a doubly infinite system of plane curves with the property that through every point of a definite region passes one curve of each system. These curves are called the u- and v-curves. The problem required the determination and further study of nets, and if possible of all such nets, with the property that in each point the tangent to the v-curve coincides with the affine normal to the u-curve and vice versa. The author shows that the solutions to the problem depend on a system of partial differential equations satisfying the Cauchy-Weierstrass conditions, which shows that the required nets depend on four arbitrary functions of one variable. No system of the net can consist of parabolas. Special attention is reserved for nets which are obtained by parallel projection of the asymptotic lines of a surface, for nets which only consist of conics, for nets which admit a one parameter and a two parameter group of affine transformations. A figure is given of a representative net of conics, consisting of two pencils of equilateral hyperbolas, with asymptotes in the directions 45° and −45°, one pencil passing through the points (1, 0), (−1, 0), the other through (0, 1), (0, −1).
D. J. Struik (Cambridge, Mass.).

Bachmann, W. K. Le théorème de Tissot et les lignes de déformation. Schweiz. Z. Vermessgswes. **37**, 253–263 (1939). [MF 831]

Mit den üblichen differentialgeometrischen Hilfsmitteln wird der bekannte Satz von Tissot bewiesen: Für jede Abbildung einer Fläche S auf eine Fläche $\bar{S}$, die nicht konform ist, gibt es genau ein orthogonales Netz auf S, dessen Bild wieder ein orthogonales Netz ist, die sogenannten Deformationslinien; in den Richtungen dieses Netzes ist die lineare Deformation extremal, wie die Betrachtung der Tissot'schen Indikatrix = Bildellipse eines infinitesimalen Kreises um einen beliebigen Punkt von S lehrt. Problem: Bestimmung der Abbildungen, die ein vorgegebenes orthogonales Netz auf S als Deformationslinien haben. *H. Samelson.*

Vincensini, Paul. Sur les réseaux isothermes sphériques. C. R. Acad. Sci. Paris **210**, 286–288 (1940). [MF 1599]

This paper remarks that the theorem given in a preceding paper [C. R. Acad. Sci. Paris **193**, 1144 (1931)] leads to a geometrical construction by means of which we can deduce from every isothermal system of lines on a sphere, or, which is the same, from every system of associated minimal surfaces, new families of associated minimal surfaces of isothermal spherical systems, and new isotropic rectilinear congruences. By a quite analogous method the author deduces from every isothermal spherical system an Appell's congruence, and from every Appell's congruence an isotropic congruence. In this way he can define Appell's surfaces (the surfaces which are normal to an Appell's congruence) by means of an analytical function which is analogous to Weierstrass' method for minimal surfaces. The geometrical interpretation of these theorems leads the author to a construction by means of which from every Appell's surface he can deduce an isothermal system of lines on a given sphere. *G. Fubini* (Princeton, N. J.).

Salini, Ugo. Trasformazioni delle reti di Voss. Atti Accad. Peloritana **41**, 141–148 (1939). [MF 1332]

Let φ, ψ be two solutions of the well-known equation

$$\frac{\partial^2\varphi}{\partial u\partial v}=\sin\varphi.$$

Starting from a theorem of Calapso, the author shows that φ and ψ determine a Vossian net in the space of four dimensions. The angles defined by

$$2\omega=\varphi+\psi,\quad 2\tau=\varphi-\psi$$

are the angle between the curves u = const., v = const. of the net and the angle between the asymptotes a, a' of the corresponding conic of Kommerel, respectively. The author determines the linear element of the net; and, in order to define the net completely, two normals n, n' have to be chosen for every point: the author chooses, in particular, the bisectors n, n' of the asymptotes a, a', and studies completely the quadratic differential forms of first order which determine the net. He next studies a second Vossian net, obtained by interchanging ω and τ, and finds some geometrical properties of this couple of nets.
G. Fubini (Princeton, N. J.).

Sauer, Robert. Fastreguläre Sechseckgewebe und fastreguläre Abbildungen. Monatsh. Math. Phys. **48**, 389–399 (1939). [MF 655]

The author considers hexagon webs (points: (u, v), lines: u = const., v = const., $u+v$ = const.) with a line element $Edu^2+2Fdudv+Gdv^2$. A hexagon is called almost regular (a.r.) if the sum of opposite sides is constant; a hexagon web is a.r. if its hexagons are a.r. Th. 1: The line element of an a.r. hexagon web has one of the three forms: (a) $du^2+dudv+dv^2$ (which we call ds^2); (b) $(u^2+uv+v^2+k)^2ds^2$; (c) $(k_1u+k_2v)^2ds^2$. The angles of intersection are constant and equal to $\pi/3$. Th. 2: A hexagon web is a.r. exactly if the large diagonals of its hexagons are equal. The case (a) is realized by three pencils of parallel straight lines in a convex region E of the Euclidean plane; in this case the hexagons are regular. The other line elements belong to elementary surfaces of revolution of non-vanishing curvature, except for $(u^2+uv+v^2)^2ds^2$, which has curvature zero. A mapping of a plane region E on a surface is called a.r. if every regular hexagon is mapped on an a.r. hexagon. These mappings are enumerated and a geometric discussion is given; they are conformal, and, in case the image surface is plane, it turns out that they are linear or else essentially the conformal mapping $w=z^3$. *M. A. Zorn.*

Bortolotti, Enea. Geometria proiettiva differenziale dei 3-tessuti di curve spaziali (terne di congruenze). Boll. Un. Mat. Ital. (2) **1**, 409–421 (1939). [MF 1310]

The author starts from a paper by Fubini [Ann. Mat. Pura Appl. (4) **16** (1937)], where, as a generalization of the nets of curves on a surface, the system of three congruences of curves in the space of three dimensions is studied. He remarks that Fubini's theory is related to the theory of projective connections and to that of anholomic surfaces. By systematical use of these theories, the author arrives at Fubini's invariants, adds (in §6) new invariants, defines

(in §7) a covariant quadric and studies the metric geometry defined by it. *G. Fubini* (Princeton, N. J.).

Humbert, Pierre. Sur les courbes planes de l'espace attaché à l'opérateur Δ_3. C. R. Acad. Sci. Paris **209**, 590–591 (1939). [MF 518]

Verfasser betrachtet eine Massbestimmung in der x, y Ebene, bei welcher der Abstand zwischen (x, y) und (x_0, y_0) durch $\{(x-x_0)^3+(y-y_0)^3\}^{\frac{1}{3}}$ gegeben ist; ferner definiert er zwei Arten der Orthogonalität wie folgt: Eine Richtung μ' ist zur Richtung μ orthogonal von erster Art, wenn $\mu'\mu^2=-1$ ist. Eine Richtung μ'' ist zur Richtung μ orthogonal von zweiter Art, wenn $\mu\mu''^2=-1$ ist. Ferner wird die Winkelhalbierende von drei einem Büschel angehörenden Geraden als diejenige Richtung definiert, deren Polargerade in bezug auf die drei gegebenen Geraden zu ihr orthogonal ist. Mit Hilfe dieses Begriffes gibt der Verfasser drei Sätze über kubische ebene Kurven: (i) Bei $y^3=3p^2x$, ist die zweite Subnormale konstant. Diese Kurve besitzt auf der x Achse zwei Brennpunkte, das heisst, zwei Punkte von denen je drei isotrope Tangenten an die Kurve gelegt werden können. (ii) Bei $y^3=3px^2$, ist die erste Subnormale konstant. Diese Kurve besitzt auf der x Achse einen Brennpunkt. (iii) Der geometrische Ort der Punkte konstanten Abstandes a vom Anfangspunkte ist gegeben durch $x^3+y^3=a^3$; die zweite Normale dieser Kurve geht immer durch den Anfangspunkt. *T. Kubota* (Sendai).

Hamburger, Hans. Beweis einer Carathéodoryschen Vermutung. Teil I. Ann. of Math. **41**, 63–86 (1940). [MF 1006]

Die Nabelpunktvermutung heisst: Jede (hinreichend differenzierbare) geschlossene Fläche vom Geschlecht Null besitzt mindestens zwei Nabelpunkte; sie wird mittels des Poincaré-schen Satzes über die Indexsumme von Kurvenscharen auf geschlossenen Flächen auf folgenden Satz zurückgeführt (dessen Beweis das Ziel der Arbeit ist): Der Index eines Nabelpunktes im Netz der Krümmungslinien ist (bei der vom Verfasser benützten Definition des Index, s.u.) nichtnegativ. Dazu wird eine spezielle Darstellung der Fläche benützt, bei der die Differentialgleichung der Krümmungslinien eine einfache Form bekommt. Der Index eines Nabels hängt im wesentlichen ab von dem zweiten Glied der Entwicklung einer gewissen Funktion, das in erster Näherung die Abweichung der Fläche von der Schmiegkugel im Nabel bestimmt; über dieses Glied werden vorläufig einschränkende Voraussetzungen gemacht. Aus einer eingehenden Untersuchung wird gefolgert, dass die Umgebung des Nabels in Sektoren zerfällt, die (im einfachsten Fall) durch gewisse Kurven, die vom Nabel ausgehen, getrennt werden. In den Punkten dieser Kurven ist (in der Parameter-Ebene) die eine Schar der Krümmungslinien tangential zu den Kreisen um den Nabel (und die andere Schar tangential zu den Radien durch den Nabel). Der Index setzt sich aus Beiträgen der Sektoren zusammen, die $+1$, 0, -1 sein können, wie die Betrachtung der Krümmungslinien in der Nähe der genannten Kurven lehrt. (Der Index ist so definiert: man konstruiere ein geschlossenes Jordanpolygon aus Netzkurvenstücken, das den singulären Punkt im Inneren enthält. Der Index ist Zahl der ausspringenden minus Zahl der einspringenden Ecken. Bei den Beiträgen der Sektoren handelt es sich um Teilstücke dieses Polygons.) Unter den geltenden, einschränkenden Voraussetzungen kann es nun Beiträge vom Wert -1 garnicht geben, woraus Index nichtnegativ folgt. Im Teil II der Arbeit sollen diese Voraussetzungen dann fallen gelassen werden. *H. Samelson* (Zürich).

Hünke, Anneliese. Über gewisse Flächen konstanter Krümmung in Räumen konstanter Krümmung. Schr. Math. Inst., Inst. Angew. Math. Univ. Berlin **4**, 259–304 (1939). [MF 1512]

This paper is a dissertation using throughout the methods of classical differential geometry and projective geometry. It is proved that the torse (envelope of one parameter family of planes) is the only type of surface in a space of constant curvature having the same total curvature as the ambient space. Known theorems on torses in euclidean space are extended to the non-euclidean. The development of non-euclidean cones on non-euclidean planes is discussed in its relationship to the space-forms. The determination of all surfaces of rotation of constant relative curvature in non-euclidean space is shown to involve at most elliptic integrals. Cases in which the integration is elementary are investigated further. Beltrami's theorem for such surfaces in euclidean space is extended to the non-euclidean. Finally Hilbert's theorem is generalized to read: With the exception of the Clifford surface there are no completely regular surfaces of constant negative relative curvature in euclidean or non-euclidean space. *J. L. Vanderslice.*

Hopf, E. Randbemerkungen zu einigen Existenzsätzen der Differentialgeometrie. Jber. Deutsch. Math. Verein. **49**, 253–255 (1940). [MF 1217]

If a surface F' is mapped pointwise on F, then the quotient ds'/ds of corresponding lineal elements at a fixed point P of F depends on the angle φ which ds includes with an initial direction at P. The expression

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{ds'}{ds}\right)^2 d\varphi=V_P^2$$

measures a mean magnification of the mapping at P. The author indicates three minimum problems in terms of V^2 of which respectively (A) the conformal mapping of two given surfaces, (B) the minimal surfaces bounded by prescribed space curves, (C) the surface embedding a given lineal element in the Euclidean space are the solutions. *H. Lewy* (Berkeley, Calif.).

Jonas, H. Intorno ad una classe notevole di cicli formati da quattro trasformazioni di Laplace nello spazio ordinario. Ann. Mat. Pura Appl. **18**, 23–50 (1939). [MF 582]

L'auteur poursuit l'étude des suites de Laplace périodiques à période 4 liées aux transformations des surfaces applicables sur une quadrique [Math. Ann. **87**, 157 (1922); Ann. Mat. Pura Appl. (4) **2**, 161 (1924–25); Math. Ann. **114**, 237 et 749 (1937); J. Reine Angew. Math. **179**, 22 (1938)]. Soit $x\,\hat{x}\,x'\,\check{x}$ un quadrilatère gauche qui décrit la suite en jeu; les diagonales $x\,x'$, $\hat{x}\,\check{x}$ engendrent deux congruences Γ, C toutes les deux W [Backes, Acad. Roy. Belgique. Bull. Cl. Sci. (5) **21**, 883 (1935)] et dont les asymptotiques des nappes focales correspondent aux développables de la suite. Si les développables de Γ, C se correspondent, les deux congruences sont R et deux réseaux focaux opposés de la suite, par exemple $(\hat{x})$ et $(\check{x})$, sont portés par la même quadrique Q. Si Q est une sphère, le couple Γ, C est stratifiable dans un sens (il existe ∞ surfaces dont les plans tangents aux points d'intersection avec $x\,x'$ passent par $\hat{x}\,\check{x}$) et il existe une surface (x^*) applicables sur le paraboloïde $z=xy$ dont les asymptotiques correspondent aux développables de la suite et le réseau permanent de la déformation à celles de Γ, C. Si on applique une transformation B_K de Bianchi à la surface (x^*), la suite subit une transformation qui lui fait

correspondre une suite de la même espèce dont les foyers opposés $\hat{x}$, $\check{x}$ sont situés sur la même sphère Q et les congruences de diagonales Γ_1, C_1 sont liées à Γ, C par une transformation asymptotique simultanée de leurs nappes focales, chaque couple Γ, Γ_1 et C, C_1 étant stratifiable conjugé; les droites qui s'appuient sur quatre diagonales Γ, C, Γ_1, C_1 ont leurs foyers sur les rayons de Γ, Γ_1 et les asymptotiques des nappes focales correspondent à celles des nappes focales de Γ, C. Pour la transformation singulier B_K à $K=\pm 1$ les rayons de Γ et C_1, ainsi que de Γ_1 et C se coupent, les points de rencontre étant des foyers d'une nouvelle congruence W. Inversement, deux systèmes de courbes tracées sur la même sphère et telles que leurs plans osculateurs non homologues se confondent deux à deux, sont ceux qui sont liés à une déformée du paraboloïde reglé et qui en représentent sur la sphère les asymptotiques par le rayon parallèle à l'axe du paraboloïde roulant sur une face ou sur l'autre.

S. Finikoff (Moscou).

Rossinski, S. Sur la déformation des congruences rectilignes avec conservation de certains systèmes spéciaux de surfaces réglées. Rec. Math. (Moscou) **5** (47), 573–636 (1939). (Russian. French summary) [MF 1349]

This paper is a recapitulation and an extension of results that were published separately. Among many special results, the first part proves such theorems as one dealing with the deformation preserving the mean ruled surfaces of the congruence. It is shown that if the lines of congruence (C) lie in the corresponding tangent planes of the surface of reference, the middle surface of C is preserved. In case the lines of C are orthogonal to these planes, the surface of reference may be any Monge surface, the mean ruled surfaces corresponding to its lines of curvature. The main problem of the second part of this paper is: for what surfaces S and what corresponding congruences C attached to S does there exist a conjugate family of isoclinic or orthogonal surfaces of this congruence which retain this property either under a general deformation or under one preserving a conjugate net of lines on S? This problem is treated in the special cases (a) when the lines of C lie in the tangent planes of S and (b) when they are orthogonal to these planes.

M. S. Knebelman (Pullman, Wash.).

Rossinski, S. Sur le problème général de la déformation des congruences avec conservation de certains systèmes spéciaux de surfaces réglées. Rec. Math. (Moscou) **6** (48), 307–330 (1939). (Russian. French summary) [MF 1360]

In this paper the author considers the problem stated in the above review, but with the lines of the congruence (C) bearing a general relation to the tangent planes of S, the deformation preserving conjugate families of isoclinic or orthogonal ruled surfaces. It is proved that in case of isoclinic families a necessary and sufficient condition for the existence of such permanent families is that the congruence be normal and that such a congruence with its lines parallel to the tangent planes of S can not exist. For a general deformation a permanent family of orthogonal surfaces can not exist, but for a deformation preserving a conjugate net on S the permanent family may be given (in a suitable coordinate system) by $dv=0$ and

$$\left\{\begin{matrix}1\\12\end{matrix}\right\}du+\left\{\begin{matrix}1\\22\end{matrix}\right\}dv=0.$$

M. S. Knebelman (Pullman, Wash.).

Vincensini, Paul. Sur les généralisations de quelques problèmes de géométrie differentielle et sur certains cycles de congruences. Acta Math. **71**, 145–174 (1939). [MF 764]

Les généralisations dont il s'agit consistent à substituer un angle $\omega=$const. à l'angle droit qui sert à définir les congruences, les réseaux orthogonaux, etc. L'auteur appelle congruences (ω) les congruences dont l'angle des plans focaux est $\omega=$const., réseaux (ω) les réseaux dont l'angle des courbes conjuguées est $\omega=$const. Toute surface S non sphérique porte deux réseaux (ω) pour chaque valeur $\omega=$const. $(0<\omega<\pi/2)$. Si C est une congruence orthogonale à un réseau (ω) (chaque rayon de C est orthogonal au plan tangent homologue de S et les développables correspondent aux courbes du réseau), les plans focaux de C sont orthogonaux aux tangentes du réseau, donc forment l'angle ω, c'est à dire, C est une congruence (ω). La congruence de Guichard généralisée ($=$deux réseaux focaux sont (ω)) est une congruence de Ribaucour dont la surface génératrice dépend de 4 fonctions arbitraires d'un argument. Les congruences orthogonaux aux réseaux focaux sont (ω), leur réseau focal commun dont deux congruences sont (ω), est un réseau de Voss généralisé ($=$les plans osculateurs des lignes du réseau forment avec le plan tangent l'angle $\omega=$const.). Le réseau admet une déformation infinitésimale conservant le réseau. Les congruences (ω) de Ribaucour ont pour génératrices les surfaces dont le rapport des rayons de courbure principaux est constant. Exemple: congruences (ω) à surface moyenne plane. Pour chaque $\omega=$const. il existe deux congruences (ω) à surface moyenne plane et à enveloppée moyenne point; elles sont nécessairement de révolution, l'axe de révolution étant la perpendiculaire abaissée du point moyen sur le plan moyen. Les congruences (ω) de Ribaucour dont les réseaux focaux se correspondent avec égalité des angles homologues, leur sont parallèles. La seconde partie du mémoire est consacrée à l'étude des transformations $T(0, \alpha)$ ($=$on tourne chaque rayon de C autour de sa parallèle issue du point fixe 0 d'un angle α [Rec. Math. (Moscou) **40**, 467 (1933)]). Si toutes les congruences transformées sont (ω), les projections de 0 sur les rayons de C partagent les segments focaux dans un rapport constant K. L'angle ω est le même pour toutes les transformées à α différent, la congruence de départ est celle d'Appell ($=$congruence normale à enveloppée moyenne point). Cycle de congruences (ω) ($=$l'ensemble des transformées à α arbitraire) est complet si les développables de tous les éléments du cycle sont réelles. Si $K>0$, le cycle est incomplet.

S. Finikoff (Moscou).

Takasu, Tsurusaburo. Neue Verallgemeinerungen der L-Minimalflächen. Monatsh. Math. Phys. **48**, 170–175 (1939). [MF 637]

Verfasser definiert $(\Delta+f)$-Mittelkugelkongruenzen und $(\Delta+f)$-L-Minimalflächen und gibt an durch welche Eigenschaften diese L-Kugelkongruenzen und L-Flächen charakterisiert sind (Verallgemeinerung eines Satzes von Kommerell).

J. Haantjes (Amsterdam).

Finikoff, Serge. Déformation projective d'une configuration (T). J. Math. Pures Appl. **18**, 405–415 (1939). [MF 1277]

A configuration (T) is a set of four congruences generated by the sides of a skew quadrilateral whose vertices describe the focal surfaces of the congruences. The purpose of the paper is to discuss the conditions on (T) in order that it

may be deformed by a projective deformation of the first order. If $M_s{}^i$, $i=1, 2, 3, 4$, the homogeneous projective coordinates of the four vertices M_s, $s=1, 2, 3, 4$, of the configuration (T) are given as functions of two variables, then these coordinates satisfy the system of Pfaff

$$dM_s=\sum_{i=1}^{4} \omega_s{}^i M_i,$$

where the $\omega_s{}^i$ are subject to certain integrability conditions and to the conditions

$$\omega_1{}^3=\omega_2{}^4=\omega_3{}^1=\omega_4{}^2=0.$$

Necessary and sufficient analytic conditions on the function $\omega_i{}^j$ that (T) be deformable are derived. The only deformable configurations are those in which the developables of the congruences generated by the pairs of opposite sides correspond directly, that is, the conjugate quadruples. However, there exist conjugate quadruples which are not projectively deformable; for example, certain quadruples of Pantazzi [Sur les couples transformable, Ann. Roumaines Math. 2, 1–33 (1935)] are not deformable. The paper concludes with showing that the configurations (T) deformable by a deformation of the second kind studied by Terracini are also conjugate quadruples and that such a configuration is deformable also by a deformation of the first order and conversely. *V. G. Grove.*

Zito, Ciro. Alcune esplicitazioni sulle trasformazioni conformi dello spazio. Atti Accad. Peloritana **41**, 86–92 (1939). [MF 1328]

Vengono esplicitati i legami analitici fra le trasformazioni conformi a cui si assoggetta una superficie S riferita alle linee di curvatura e le trasformazioni proiettive, in cui si traducono le suddette trasformazioni conformi, quando la superficie S si assoggetta a trasformazioni di Lie. Si caratterizza il sottogruppo proiettive Γ, che corrisponde alle trasformazioni conformi dello spazio e si ritrova la trasformazione conforme singolare che, nel campo analitico, non rientra nel teorema di Liouville. *Author's abstract.*

Salini, Ugo. Osservazioni sulle normali ad una superficie di uno spazio a quattro dimensioni. Atti Accad. Peloritana **41**, 52–54 (1939). [MF 1327]

Let Σ be a surface of two dimensions in a space S of four dimensions. Let us suppose that through every point A of Σ there pass two straight lines n_1, n_2 which are normal to each other and also to Σ. Suppose further that there is a net of curves u, v on Σ such that the lines n_i $(i=1, 2)$ generate a developable surface if A moves on a line $u=\text{const.}$ or on a line $v=\text{const.}$ of the net. This is possible only for particular surfaces Σ, which were studied by Guichard [Ann. École Norm. 1897, 1898, 1903]. For these surfaces we can define the radii r_1, r_2 of curvature for the lines n_1 and the radii R_1, R_2 for the lines n_2. If we know the equations of the surface Σ, integrations are necessary to obtain the four radii r_i, R_i. But the complex curvatures

$$\frac{1}{\rho_i}=\frac{1}{r_i}+\frac{\sqrt{-1}}{R_i}, \qquad i=1, 2,$$

have the property that every homogeneous function of degree zero of $1/\rho_1$, $1/\rho_2$ may be calculated by differentiations. An analogous theorem is given for a couple of surfaces Σ, which are equivalent with respect to the group of conformal transformations in the space S. The development and the demonstrations of these theorems are to be given in another paper. *G. Fubini* (Princeton, N. J.).

Kanitani, Jôyo. On a such surface with two systems of asymptotic curves that its every tangent plane touches it along a curve. Tensor **2**, 29–31 (1939). (Japanese) [MF 1366]

The fundamental equations are derived for a surface with two systems of asymptotic curves in a conformally connected space of three dimensions such that its every tangent plane touches it along a curve. Such a surface can not exist in the Riemannian space. It is also shown that the surface has many properties similar to those of a regular surface. *A. Kawaguchi* (Sapporo).

Sasaki, Shigeo. A new proof of the theorem of J. A. Schouten on an umbilic surface in a Riemannian space. Tensor **2**, 25–29 (1939). (Japanese) [MF 1365]

The conditions for the existence of an umbilic hypersurface with a given direction at any point are investigated, making use of the fundamental derived equations and the Gauss-Codazzi's conditions for integrability in terms of the normal coordinates. Then as the required conditions the equations

$$C_{ijkl}X^i_{.a}X^j_{.b}X^k_{.c}N^l=0$$

are obtained, where C_{ijkl} denotes the conformal curvature tensor of Weyl,

$$X^i=X^i(\dot{x}^1, \dot{x}^2, \cdots, \dot{x}^{n-1})$$

are the equations of the required hypersurface, $X^i_{.a}=\partial X^i/\partial x^a$ and N^l is the normal unit vector of the hypersurface. From this condition follows immediately the theorem of Schouten [Math. Z. 11, 59–88 (1921)]: In order that in an n-dimensional Riemannian space there always exist an umbilic hypersurface passing through any one given point and having an arbitrarily given tangent hyperplane at that point, it is necessary and sufficient that the space be conformally flat. *A. Kawaguchi* (Sapporo).

Calapso, Renato. Sui sistemi di geodetiche generalizzate appartenenti ad una superficie di un S_3. Atti Accad. Peloritana **41**, 128–137 (1939). [MF 1331]

Referred to its asymptotic curves the differential equations of a surface V_2 in S_3 may be written in the form

$$\frac{\partial^2 x}{\partial u^2}=a\frac{\partial x}{\partial u}+b\frac{\partial x}{\partial v}, \quad \frac{\partial^2 x}{\partial v^2}=p\frac{\partial x}{\partial u}+q\frac{\partial x}{\partial v}.$$

The author calls a curve $u=u(t)$, $v=v(t)$ a generalized geodesic g on V_2 if $u(t)$ and $v(t)$ satisfy the equation

$$\frac{du}{dt}\frac{d^2v}{dt^2}-\frac{dv}{dt}\frac{d^2u}{dt^2}=(a+2\beta)\left(\frac{du}{dt}\right)^2\frac{dv}{dt} -(g+2\alpha)\frac{du}{dt}\left(\frac{dv}{dt}\right)^2-M\left(\frac{du}{dt}\right)^3+N\left(\frac{dv}{dt}\right)^3,$$

wherein α, β, M, N are arbitrary functions. A set of curves C defined by $k\,du-ch\,dv=0$ $(c=\text{const.})$ and the conjugate set C' defined by $k\,du+ch\,dv=0$ are also considered. The osculating plane to C (C') intersects the osculating plane to g which is tangent to C' (C) in a line l (l'). The paper concerns itself with these lines and the axis $\bar{l}$ of the system (g) of generalized geodesics. In particular, analytic conditions that l, l', $\bar{l}$ lie in a plane for every value of c are derived. A unique intrinsic system (g) of generalized geodesics is derived. *V. G. Grove* (East Lansing, Mich.).

Castoldi, Luigi. Su alcuni casi in cui il trasporto per parallelismo di un vettore superficiale è un trasporto parallelo nello spazio ambiente. Period. Mat. **19**, 260–264 (1939). [MF 901]

Three cases are discussed.

I. What is the most general surface Σ such that for any two points P_1 and P_2 on it (1) there exists at least one surface vector v_1 at P_1 which, if transported by parallelism of Levi-Civita along an arbitrary curve on Σ passing through P_1 and P_2, leads to a surface vector v_2 at P_2 parallel to v_1 in the ordinary sense; (2) there exists at least one curve γ on Σ connecting P_1 and P_2 and at least one surface vector v_1 at P_1 such that transportation by parallelism of Levi-Civita of v_1 along γ to P_2 is also parallel transportation in the ordinary sense?

II. Given an arbitrary surface, what is the most general curve γ such that, for two points P_1 and P_2 on it, every surface vector v_1 at P_1, transported by parallelism of Levi-Civita along γ, leads to a surface vector v_2 at P_2 parallel to v_1 in the ordinary sense?

The answer to I_1 is the cylindrical surfaces, to I_2 is any surface, to II is the plane asymptotic curves, if they exist.

D. J. Struik (Cambridge, Mass.).

Yano, Kentaro. Sur les équations de Gauss dans la géométrie conforme des espaces de Riemann. Proc. Imp. Acad., Tokyo **15**, 247–252 (1939). [MF 1142]

Conformal equations corresponding to the equations of Gauss for a sub-space V_m of a Riemannian space V_n are developed. Several consequences are noted; for example, if, in a Riemannian space V_n, which is conformal to a Euclidean space, there is a sub-space V_m which is totally umbilical, then V_m necessarily is conformal to a Euclidean space $(n>m>3)$.

E. F. Beckenbach (Houston, Tex.).

Yano, Kentaro. Sur les équations de Codazzi dans la géométrie conforme des espaces de Riemann. Proc. Imp. Acad., Tokyo **15**, 340–344 (1939). [MF 1269]

The work of the preceding article is continued, equations of Codazzi now being determined for conformal geometry of Riemann spaces. The idea is to obtain equations analogous to the Gauss and Codazzi equations when the Riemann-Christoffel tensor is replaced by Weyl's conformal curvature tensor.

E. F. Beckenbach (Houston, Tex.).

Sasaki, Shigeo. On the theory of surfaces in a curved conformal space. Tensor **2**, 52–53 (1939). (Japanese) [MF 1373]

As the fundamental vectors of a hypersurface

$$X^k = X^k(x^{\dot 1}, x^{\dot 2}, \cdots, x^{\dot n - \dot 1}), \quad i, j, k = 1, 2, \cdots, n,$$

in a space with a conformal connection, the author takes n spheres

$$\mathfrak{B}_p{}^A \equiv \frac{\partial \mathfrak{B}^A}{\partial x^p} + \Gamma^A_{B\gamma}\mathfrak{B}^B \frac{\partial X^\gamma}{\partial x^p} + \left(\frac{1}{n}\frac{\partial X^0}{\partial x^p} - \frac{1}{n-1}\delta_p{}^{\dot 0}\right)\mathfrak{B}^A,$$

$$\mathfrak{B}_{\dot\infty}{}^A \equiv -G^{ab}\left(\frac{\partial \mathfrak{B}_a{}^A}{\partial x^b} + \Gamma^A_{Bk}\mathfrak{B}_a{}^B \frac{\partial X^k}{\partial x^b} - \Gamma^r_{ab}\mathfrak{B}_r{}^A\right) + \lambda \mathfrak{B}_{\dot 0}{}^A,$$

$$A, B = 0, 1, \cdots, n, \infty; \quad \gamma = 0, 1, \cdots, n; \quad \dot p = \dot 0, \dot 1, \cdots, \dot n - \dot 1;$$
$$a, b, r = \dot 1, \cdots, \dot n - \dot 1,$$

and a unit sphere $\mathfrak{N}^A$ which is orthogonal to $\mathfrak{B}_p{}^A$ and $\mathfrak{B}_{\dot\infty}{}^A$, where

$$\Delta^{2(n-1)} \equiv \det|(ab)|, \quad (ab) \equiv \frac{1}{n^2} G_{ij}\frac{\partial X^i}{\partial x^a}\frac{\partial X^j}{\partial x^b}, \quad \mathfrak{B}^A \equiv \Delta^{-1}\delta_0{}^A,$$

$$G_{ab} \equiv \mathfrak{G}_{AB}\mathfrak{B}_a{}^A\mathfrak{B}_b{}^B, \quad \lambda \equiv \tfrac{1}{2}\mathfrak{G}_{AB}\mathfrak{B}^A_{\cdot\dot\infty}\mathfrak{B}^B_{\cdot\dot\infty}, \quad \mathfrak{B}^A_{\cdot\dot\infty} \equiv \mathfrak{B}_{\dot\infty}{}^A - \lambda\mathfrak{B}_{\dot 0}{}^A,$$

and Γ^r_{ab} are the parameters of the conformal connection formed by G_{ab} in the hypersurface. The fundamental equations are then given by

$$\begin{aligned} \mathfrak{B}^A_{P,Q} &= b^R_{PQ}\mathfrak{B}_R{}^A + b_{PQ}\mathfrak{N}^A, \\ \mathfrak{N}^A_{\cdot Q} &= b_Q{}^R\mathfrak{B}_R{}^A + b_Q\mathfrak{N}^A, \end{aligned} \qquad P, Q, R = 0, 1, \cdots, n, \infty,$$

making use of the complete conformal derivatives. It follows from the properties of the complete conformal derivatives that b's can be expressed by b_{ab}, $b_{\dot\infty a}$, b^0_{ab}.

A. Kawaguchi (Sapporo).

Mutô, Yosio. On the equations of circles in a space with a conformal connection. Tensor **2**, 50–52 (1939). (Japanese) [MF 1372]

At any point in a conformally connected space X_n with coordinates x^i we can consider an $(n+1)$-dimensional projective space P_{n+1} with coordinates u^P and a hypersurface of second order Q_n contained in P_{n+1}. Any curve in X_n is mapped into a curve $u^P(\sigma, t)$ in Q_n at any point $x^i(t)$ on the given curve by the equations

$$-\frac{\partial u^P}{\partial \sigma} + \frac{\partial u^P}{\partial t} + \Pi^P_{Qr}u^Q\dot x^r = 0,$$

where Π^P_{Qr} are the parameters of the conformal connection. Making use of these equations the author derives the equations of a circle in X_n whose image in Q_n is the section of Q_n by a plane in P_{n+1}:

$$\begin{aligned} &\frac{d}{ds}(x''^i + \Pi^i_{jk}x'^jx'^k) + \Pi^i_{jk}(x''^j + \Pi^j_{ab}x'^ax'^b)x'^k + \Pi^i_{0k}x'^k \\ &\qquad - \Pi^0_{jk}x'^jx'^kx'^i + G_{jk}(x''^j + \Pi^j_{ab}x'^ax'^b)(x''^k + \Pi^k_{cd}x'^cx'^d)x'^i = 0, \end{aligned}$$

where $ds^2 = G_{ij}dx^idx^j$.

A. Kawaguchi (Sapporo).

Mutô, Yosio. On some properties of hypersurfaces in a conformally connected manifold. Proc. Phys.-Math. Soc. Japan **21**, 615–625 (1939). [MF 1126]

A conformal connection is induced in a hypersurface of a conformally connected space by choosing a point field (conformal vector) in the associated projective $(n+1)$-spaces and "projecting" the connection into the hypersurface. The procedure is quite analogous to that used in spaces with an affine or projective connection. The components of the induced connection are computed in terms of the original connection and the chosen vector. Umbilical point on a hypersurface is defined in a conformally invariant way and several theorems are proved relating umbilical points and generalized circles.

J. L. Vanderslice (Bethlehem, Pa.).

Haantjes, J. Eine Charakterisierung der konformeuklidischen Räume. Nederl. Akad. Wetensch., Proc. **43**, 91–94 (1940). [MF 1238]

The present paper generalizes a previous result of Haantjes and Wrona on even dimensional conformally flat spaces. The new theorem applies to any dimension above three. Let $m_i>1$ be arbitrary but fixed dimensionalities for a set of $p>1$ mutually perpendicular m-directions at a point. Let k_i be the corresponding scalar curvatures. Then

a necessary and sufficient condition that a conformal space of $n>3$ dimensions be flat is that $\sum_1{}^p m_i k_i$ at every point be independent of the particular m-directions chosen.

J. L. Vanderslice (Bethlehem, Pa.).

Michal, A. D. and Mewborn, A. B. Abstract flat projective geometry. Proc. Nat. Acad. Sci. U. S. A. **25**, 440–443 (1939). [MF 892]

Introducing a real valued gauge variable x^0, a flat projective connection $\Pi(X, Y, Z)$ is defined in a geometric space with allowable coordinates in a Banach space B under the assumption of existence of projective coordinate systems, and some theorems are stated. X, Y and Z represent here every one element in the second Banach space B_1 of couples (x, x^0), where x is in B. Then an answer is given to the converse problem, that is the principal result in this paper: What assumptions on the parameters of connection $\Pi(X, Y, Z)$ with arguments and values in B_1 are sufficient for the existence of projective coordinate systems, when the geometric space is a Hausdorff topological space with allowable $K^{(3)}$ coordinates in B? There is no proof for all results.

A. Kawaguchi (Sapporo).

Hokari, Shisanji. Geometry of connections in an abstract space. Tensor **2**, 7–13 (1939). (Japanese) [MF 1363]

A sketch of the theory of linear connections in an abstract space. After the definition of a contra- or covariant vector under a group of abstract coordinate transformations, the three postulates for the linear connection and the fundamental theorems on the torsion and curvature tensors are stated. The parallelism along a curve, the projective theory of paths and the Bianchi's identity are also explained. The author touches on abstract Riemannian geometry too.

A. Kawaguchi (Sapporo).

Kosambi, D. D. The tensor analysis of partial differential equations. Tensor **2**, 36–39 (1939). (Japanese) [MF 1368]

The author determines by his own method of variations the parameters of a connection $\gamma^i_{\alpha j}$, $\Gamma^\gamma_{\alpha\beta}$ under the transformations of coordinates x^i and parameters u^α from a system of partial differential equations of second order

$$\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}+H^i_{\alpha\beta}\left(u^\gamma, x^j, \frac{\partial x^j}{\partial u^\gamma}\right)=0.$$

This connection is not of an affine type but of a projective type. All differential invariants of the system can be derived by means of this connection. *A. Kawaguchi* (Sapporo).

Kawaguchi, Akitsugu. Views on higher order geometry of connections. II. Tensor **2**, 39–45 (1939). (Japanese) [MF 1369]

By higher order geometry we understand the geometry in those spaces whose geometrical elements are surface elements of several dimensions and of higher order:

$$x^i, \frac{\partial x^i}{\partial u^\alpha}, \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \cdots, \frac{\partial^M x^i}{\partial u^{\alpha_1}\cdots\partial u^{\alpha_M}},$$
$$i=1, 2, \cdots, n; \quad \alpha, \beta=1, 2, \cdots, m.$$

The fundamental notion of "base connections" is introduced, without which it seems impossible to establish the higher order geometry of connections parallel to the ordinary geometry of connections. The base connections can be defined by two postulates. An example can be found in the theory of connections of a system of partial differential equations of second order

$$\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}+H^i_{\alpha\beta}\left(u^\gamma, x^j, \frac{\partial x^j}{\partial u^\gamma}\right)=0,$$

and the author shows the relation between the theories of Bortolotti [Atti Accad. Naz. Lincei. Rend. **23**, 16–21, 104–110, 175–180 (1936)] and of Kosambi [cf. above review]. Then some remarks on the geometrical theories of the ordinary or partial differential equations and on the theory of extensors follow. In conclusion the author stresses the importance of the theory of subspaces in the higher order geometry and the rheonomic geometry or, more generally, the so-called Kawaguchi-Hokari geometry in a higher order space.

S. Hokari (Sapporo).

Hombu, Hitoshi. Theory of paths of higher order and its application. Tensor **2**, 32–36 (1939). (Japanese) [MF 1367]

The application of the projective theory of a system of ordinary differential equations of higher order established by the author [Jap. J. Math. **15**, 139–196 (1938); J. Fac. Sci. Hokkaido Imp. Univ. **7**, 35–94 (1938)] enables us to determine an affine connection in a Kawaguchi space of higher order. Moreover, by use of the differentiation

$$\frac{D}{dt}f=\sum_{\nu=0}^{m-1}\frac{\partial f}{\partial x^{(\nu)i}}x^{(\nu+1)i}-\frac{\partial f}{\partial x^{(m)i}}H^i, \quad x^{(\nu)i}=\frac{d^\nu x^i}{dt^\nu},$$

along the paths of higher order

$$\overset{m-1}{\mathfrak{E}}_i\equiv f_{ij}(x^{(m+1)j}+H^j)=0,$$

where $\overset{m-1}{\mathfrak{E}}_i$ denotes an intrinsic Synge vector, or by means of the projective invariants $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_{m-1}$ of the system of paths, the metric connection in a Kawaguchi space of order m introduced by Kawaguchi [Proc. Imp. Acad. Tokio **13**, 237–240 (1937)], whose parameters depend on a curve element of order $2m-1$, can be modified so that the parameters of the connection will be dependent of a curve element of order m.

A. Kawaguchi (Sapporo).

Yano, Kentarô. Theory of non-holonomic spaces. Tensor **2**, 13–24 (1939). (Japanese) [MF 1364]

The first part concerns the history of the development of the theory of non-holonomic spaces. In the second part an outline is given of the theory and of its applications to the unified theory of Einstein and Mayer. *A. Kawaguchi.*

Davies, E. T. Lie derivation in generalized metric spaces. Ann. Mat. Pura Appl. **18**, 261–274 (1939). [MF 922]

The first part of this paper compares the fundamental invariants of a Finsler space obtained by using (1) Berwald's and (2) Cartan's affine connection. The second part considers an infinitesimal transformation $\bar{x}^i=x^i+v^i(x)dt$ in the space and what happens to the components of a geometrical object under this transformation. The result depends on the interpretation of this transformation; we may have a field of objects and desire to obtain the change in the components from the point x to the point $\bar{x}$; or the object may be carried by parallel displacement from x to $\bar{x}$. We may also look upon our transformation as a mapping of the space upon itself and then an object at x is mapped into an object at $\bar{x}$ by its law of transformation. The paper gives the expressions for the differences in the various changes in

terms of covariant derivatives and the geometrical meaning of the vanishing of some of them. *M. S. Knebelman.*

Berwald, L. Über die n-dimensionalen Cartanschen Räume und eine Normalform der zweiten Variation eines $(n-1)$-fachen Oberflächenintegrals. Acta Math. **71**, 191–248 (1939). [MF 765]

This paper is dedicated to Elie Cartan on the occasion of his 70th birthday. It is largely expository in character, centers quite naturally around ideas initiated by Cartan and is developed in the Princeton tradition. The paper is divided into three parts. The first part deals with an invariant theory in which the point (as an underlying space element) is replaced by an oriented hypersurface element, that is, the figure consisting of a point and an oriented hyperplane passing through the point. Going out from this idea a number of assumptions are made as the basis of the analytical work. The following first two assumptions are typical and suggestive of the subsequent treatment. (1) The square of the distance of an arbitrary point (x) and an arbitrary infinitely near point $(x+dx)$ with respect to a hypersurface element at (x), whose position is arbitrary, is given by a positive definite quadratic differential form $g_{ik}dx^idx^k$ having coefficients $g_{ik}=g_{ki}$ which depend only on the hypersurface element (that is, the g_{ik} depend on the coordinates $x^1, \cdots, x^n$ of the point (x) and n parameters $u_1, \cdots, u_n$ not all of which are equal to zero). (2) Let (X) be an arbitrary vector at an arbitrary hypersurface element (x, u) and $(x+dx, u+du)$ an arbitrary hypersurface element infinitely near to (x, u). Then the vector $(X+dX)$ at the hypersurface element $(x+dx, u+du)$ is said to be parallel to the vector (X) at the hypersurface element (x, u) if

$$dX^i=-X^k[\Gamma^i_{kh}(x, u)dx^h-C_k^{ih}(x, u)du_h],$$

to within quantities of order higher than the first, where the $\Gamma_{kh}{}^i$ and $C_k{}^{ih}$ are given functions of the x^i and u_i.

Assuming that the length of an arbitrary vector is unaltered by parallel displacement relations between the g's, the Γ's and the C's are obtained. A function $L(x, u)$, called the fundamental function, is then introduced and the integral of this function gives (by definition) the $n-1$ dimensional volume of a hypersurface. By differentiating the function L^2 one obtains quantities which may be taken as the components of the above tensor g, and by imposing further assumptions it is shown that the equations defining the parallel displacement of vectors assume the precise form of these equations in an affine space. It is shown moreover that when a certain tensor H of the second order has a nonvanishing determinant the equations for the parallel displacement of vectors have a unique determination in terms of the fundamental function. Such spaces are said to be regular. The remainder of the first part of the paper is devoted among other things to the definition of the so-called torsion tensor and various curvature tensors and includes the interpretation of the vanishing of certain of these tensors.

The second part of the paper is devoted to the theory of hypersurfaces in a regular space. The fundamental forms of the hypersurface are defined. Also the theory of covariant differentiation, Bianchi identities, Gauss-Codazzi equations, torsion, curvature, etc. are discussed. The third part is concerned with the variation of the integral of the fundamental function over a hypersurface in a regular space. It is shown that the extremal hypersurfaces of the variation problem are hypersurfaces whose mean curvature is zero (minimal hypersurfaces). The second variation is then considered, the domain of integration being an extremal hypersurface with fixed $n-2$ dimensional boundary, and a normal form for this variation is obtained which is identical with one previously obtained by Koschmieder. The paper closes with an appendix devoted to various details of the calculations involved in the variation problem.

T. Y. Thomas (Los Angeles, Calif.).

de Mira Fernandes, Aureliano. Derivate tensoriali simmetriche. Ist. Lombardo, Rend. **72**, 160–164 (1939). [MF 948]

The author, in two previous notes [Atti Accad. Naz. Lincei. Rend. (6ª) 9], has discussed some tensors associated with a vectorial ennuple in a Riemannian manifold. Here he shows how a symmetrical second derivative of a vector field can be defined, and gives conditions under which this tensor can be the derivative of a bivalent tensor field or the second derivative of a vector field. *D. J. Struik.*

Sibata, Takasi. Spinor calculus. II. J. Sci. Hirosima Univ. Ser. A **9**, 165–193 (1939). [MF 671]

If the quantities $\gamma_{\lambda\cdot B}^{A}$ $(A, B, \cdots=1, \cdots, 4; \lambda, \mu=1, \cdots, 5)$ satisfy the equation $\gamma_{(\lambda}\gamma_{\mu)}=g_{\lambda\mu}$ $(g_{5i}=0, g_{55}=\pm 1, i=1, \cdots, 4)$, there exists a matrix $D^{\bar{A}}_{\cdot B}$ for which $\bar{\gamma}_\lambda=\epsilon D\gamma_\lambda D^{-1}$, where $\epsilon=$sign det $(g_{\lambda\kappa})$. The author considers several bases for spin-vectors φ^A. If ψ^A is any spin-vector, then, in general, a basis is formed by $\psi, \gamma_5\psi, D^{-1}\psi, \gamma_5D^{-1}\psi$; other bases are formed by $\psi, \gamma_5\psi, \gamma_{[i}\gamma_{j]}\psi$; $\gamma_i\psi, \gamma_i\gamma_5\psi$, both with real or pure imaginary coefficients. The relations between these bases are obtained. The author gives the general solution of the following equation for v^λ: $v^\lambda\gamma_\lambda\psi=v\psi$. In part II a covariant differentiation of spinors is introduced. The conditions of integrability of the fundamental equation $\nabla_i\psi=\sum_i\psi$ are investigated. *J. Haantjes* (Amsterdam).

ANALYSIS

Differential Equations, Operational Calculus

Gorélik, G. Rétroaction retardée. Acad. Sci. U.S.S.R. J. Phys. **1**, 465–470 (1939). [MF 1450]

The author studies and interprets physically special equations of type $y''(t)+\omega^2y(t)=\lambda f(y(t), y'(t), y(\tau), y'(\tau))$, where λ is a small parameter, $\omega=$const., and $\tau=t-h<t$, $h=$const. By the transformation $y(t)=u(t)\cos\omega t+v(t)\sin\omega t$, $y'(t)=-\omega u(t)\sin\omega t+\omega v(t)\cos\omega t$, the equation reduces to two expressions for u and v; the author then replaces the right-hand members by their time-averages. No claims to mathematical rigor are made. *W. Feller.*

Nagumo, Mitio. Über das Verhalten der Integrale von $\lambda y''+f(x, y, y', \lambda)=0$ für $\lambda\to 0$. Proc. Phys.-Math. Soc. Japan **21**, 529–534 (1939). [MF 1102]

Suppose $f(x, y, z, \lambda)$ is of class C' in a region $\mathfrak{B}[0\leqq x\leqq l;$ $|y-Y(x)|\leqq a(x)$; $|z-Y'(x)|\leqq pe^{-cx}+b(x)$; $|\lambda|\leqq h]$, where l, p, c and h are positive constants, $a(x)$ and $b(x)$ are positive continuous functions on $0\leqq x\leqq l$, and $Y(x)$ is a function of class C'' which satisfies the differential equation $f(x, Y, Y', 0)=0$ on this interval; moreover, there are positive constants K, L, Λ and M such that $|f_y|\leqq K, f_z\geqq L$, $|f_\lambda|\leqq\Lambda$ on $\mathfrak{B}$ and $|Y'(x)|\leqq M$ on $0\leqq x\leqq l$. Under these conditions the author shows that there is a $\lambda_1>0$ such

that if $0<\lambda<\lambda_1$ and $y(x)$ is a solution of the differential equation $\lambda y''+f(x, y, y', \lambda)=0$ satisfying $y(0)=Y(0)$, $|y'(0)-Y'(0)|\leqq p$, then

$$|y(x)-Y(x)|<\lambda\left(\frac{\Lambda+M}{K}+\frac{p}{L}\right)e^{(K/L)x}$$

on $0\leqq x\leqq l$. Further associated results are obtained by the use of this inequality and a slight modification of the usual method of successive approximations. *W. T. Reid.*

Chiellini, Armando. Sull'equazione differenziale lineare soddisfatta dal prodotto di integrali di un'equazione differenziale lineare del 2° ordine. Boll. Un. Mat. Ital. (2) 1, 435–436 (1939). [MF 1313]

This note indicates how the result proved by Palamà [Boll. Un. Mat. Ital. (2) 1, 230–235 (1939)] is an immediate consequence of known results. *W. T. Reid.*

Chiellini, Armando. Sull'integrazione dell'equazione differenziale $Y^{(n)}+P_n(x)Y=0$. Boll. Un. Mat. Ital. (2) 1, 426–434 (1939). [MF 1312]

The author deduces, by means of the theory of invariants, a necessary and sufficient condition for the linear differential equation $Y^{(n)}+P_n(x)Y=0$ to be integrable by quadratures. The difference in the functional character of this equation according as $n=2$, 3 or $n>3$ is also discussed. *W. T. Reid* (Chicago, Ill.).

Fréchet, Maurice. Sur l'intégration d'un système canonique d'équations différentielles linéaires à coefficients discontinus. Proc. Benares Math. Soc. 1, 1–14 (1939). [MF 1526]

This paper studies the system of first order linear differential equations

$$\frac{dx_k}{dt}=\sum_{i=1}^{r} a_{ik}(t)x_i+F_k(t), \qquad k=1, 2, \cdots, r,$$

and the related system of integral equations

$$x_k(t)=x_k(s)+\int_s^t \sum_{i=1}^{r} a_{ik}(z)x_i(z)dz+\int_s^t F_k(z)dz.$$

By way of introduction, the well-known fundamental existence and uniqueness theorems for the case where the coefficients are assumed continuous and the case where these coefficients are summable in the Lebesque sense are established by methods of successive approximations. Fréchet then changes the integral system to the form

$$x_k(t)=x_k(s)+\int_s^t \sum_{i=1}^{r} x_i(z)d\left[\int_s^z a_{ik}(z_1)dz_1\right]+\int_s^t F_k(z)dz,$$

and observes that the functions $\int_s^z a_{ik}(z_1)dz$, $\int_s^t F_k(z)dz$ are absolutely continuous, hence continuous and of bounded variation, on any interval S': $s\leqq t\leqq T$, where $a_{ik}(t)$ and $F_k(t)$ are summable. He then considers the Stieljes integral system

$$x_k(t)=x_k(s)+\sum_{i=1}^{r}\int_s^t x_i(z)dA_{ik}(z)+B_k(t), \quad k=1, \cdots, r,$$

where $A_{ik}(t)$ and $B_k(t)$, $k=1, \cdots, r$, are continuous and of bounded variation on S: $s\leqq t\leqq T$, and where all of these functions vanish at $t=s$. The same existence and uniqueness theorems are established for this system as were established in the special case where the coefficients were summable in the Lebesgue sense. A modified form of the method of successive approximations is used to prove the existence theorem. *W. M. Whyburn* (Los Angeles, Calif.).

Kakeya, Sôichi and Tsuji, Masatsugu. On the measure of section of the integral curves. Jap. J. Math. 16, 71–78 (1939). [MF 1106]

Tsuji, Masatsugu. On Lindelöf's theorem in the theory of differential equations. Jap. J. Math. 16, 149–161 (1939). [MF 1108]

In the second paper, the author extends a theorem of Lindelöf [J. Math. Pures Appl. (5) 6 (1900)] for differential systems

$$y_i'=f_i(x, y_1, \cdots, y_n), \qquad i=1, \cdots, n,$$

to the case in which the functions f_i are continuous on $I: 0\leqq x\leqq a$, $|y_i|\leqq b$ $(i=1, \cdots, n)$ and satisfy a Lipschitz condition with respect to $(y_1, \cdots, y_n)$ on this region. By means of this theorem he then proves the following result which contains those of the paper by Kakeya and Tsuji as special cases: Let M be a constant such that $|f_i|\leqq M$, $(i=1, \cdots, n)$ on I and set $\alpha=\min(a, b/M)$. Then the totality of integral curves of the above system issuing from the origin $(x, y_1, \cdots, y_n)=(0, 0, \cdots, 0)$ intersects an arbitrary plane $x=x_1$ $(0<x_1<\alpha)$ in a point set of n-dimensional measure zero whenever the following conditions are satisfied: (1) for each x_0 on $0<x<a$ the functions f_i satisfy on $x_0\leqq x\leqq a$, $|y_i|\leqq b$ a Lipschitz condition in $(y_1, \cdots, y_n)$ with constant $K=K(x_0)$, which may tend to infinity in any manner as $x_0\to 0$; (2) for fixed $x>0$, except possibly on a set of measure zero, the condition $\sum_{i=1}^n \partial f_i/\partial y_i\leqq n/x$ holds almost everywhere in $(y_1, \cdots, y_n)$-space. *W. T. Reid.*

Drach, Jules. Sur un problème relatif aux formes différentielles linéaires. C. R. Acad. Sci. Paris 210, 125–128 (1940). [MF 1288]

The author deals with a differential form $Rd\varphi-Pdx-Qdy$, where R, S and Q are polynomials of degree m in φ with coefficients depending on x and y. He gives a method for determining these coefficients in such a way that the form admits an integrating factor. *E. Rothe.*

Germay, R. H. J. Remarque sur les systèmes complètement intégrables d'équations aux différentielles totales. Mathesis 53, 286–289 (1939). [MF 1273]

It is known that the integrals of the completely integrable system

$$dz_i=\sum_{k=1}^{k=n} a_{ik}(x_1, x_2, \cdots, x_n; z_1, \cdots, z_m)dx_k, \quad i=1, 2, \cdots, m,$$

$$z_i(x_1^0, \cdots, x_n^0)=z_i^0$$

may be implicitly defined by the equations $U_i(x_1, \cdots, x_n; z_1, \cdots, z_m)=z_i^0$, in which the U_i are the solutions of the complete jacobian system

$$\frac{\partial U_i}{\partial x_k}+a_{1k}\frac{\partial U_i}{\partial z_1}+\cdots+a_{mk}\frac{\partial U_i}{\partial z_m}=0, \quad k=1, \cdots, n,$$

$$U_i(x_1^0, \cdots, x_n^0, z_1, \cdots, z_m)=z_i.$$

The author shows that this fact may be established by an argument similar to the one which is used for the reduction of a partial differential equation $\partial z/\partial x_1=f(x_1, \cdots, x_n, z, \partial z/\partial x_2, \cdots, \partial z/\partial x_m)$ to a partial differential equation of the

form

$$\frac{\partial V}{\partial x_1}+\frac{\partial V}{\partial z}f\left(x_1, \cdots, x_n, z, -\frac{\partial V/\partial x_2}{\partial V/\partial z}, \cdots, -\frac{\partial V/\partial x_n}{\partial V/\partial z}\right)=0.$$

E. Rothe (Oskaloosa, Iowa).

Kamke, E. Über die definiten selbstadjungierten Eigenwertaufgaben bei gewöhnlichen linearen Differentialgleichungen. I. Math. Z. **45**, 759–787 (1939). [MF 1414]

This paper treats self-adjoint boundary value problems of the form $L(y)=\lambda g(x)y$, $U_\mu(y)=0$ $(\mu=1, \cdots, n; a\leqq x\leqq b)$, where $g(x)\not\equiv 0$ on ab, $L(y)$ is a linear differential operator of even order n, and the $U_\mu(y)$ are n independent linear forms in the end-values of $y, y', \cdots, y^{(n-1)}$ at a and b. The problem is supposed "definite" in the sense that $\int_a^b yL(y)dx \geqq 0$ for arbitrary functions $y(x)$ of class $C^{(n)}$ on ab which satisfy $U_\mu(y)=0$ $(\mu=1, \cdots, n)$; however, the function $g(x)$ is not restricted as to sign on this interval. It is also supposed that $\lambda=0$ is not a characteristic value; however, in case $g(x)>0$ [or $g(x)<0$] throughout ab it is readily seen that this last hypothesis can be omitted. Under these conditions the characteristic values, which are all real, are infinite in number; in the polar case where $g(x)$ changes sign on ab the problem has infinitely many positive, and also infinitely many negative, characteristic values. The author proves an expansion theorem in terms of the characteristic functions, and shows that the characteristic values are determined by the maximum-minimum principle of Courant. Finally, the approximation of the characteristic values by the Ritz-Galerkin method is considered.

W. T. Reid (Chicago, Ill.).

Leray, Jean. Discussion d'un problème de Dirichlet. J. Math. Pures Appl. **18**, 249–284 (1939). [MF 806]

Dans un mémoire antérieur [J. Math. Pures Appl. **17**, 89–104 (1938)], l'auteur a trouvé dans certains cas une limite supérieure de p^2+q^2 quand z satisfait dans un domaine Δ à une équation (1) $f(r, s, t, p, q, x, y, z)=0$ non linéaire et du type elliptique et prend une suite de valeurs données sur la frontière Δ' de Δ (problème de Dirichlet). Ici l'auteur tire parti de ces résultats pour discuter le problème de Dirichlet dans certains cas où, pour x, y, z, p, q fixes, (1) représente dans l'espace (r, s, t) une surface dont la courbe à l'infini n'a aucune tangente commune avec $rt-s^2=0$; certaines conditions de régularité sont imposées à f. Les raisonnements et les énoncés font intervenir un échange des rôles de z et de y. L'auteur cite des cas généraux où le problème de Dirichlet est impossible, et d'autres où il est possible et jouit de propriétés que l'auteur exprime par la locution: "problème bien posé." Ces résultats sont appliqués à des cas où f est linéaire par rapport à (r, s, t), et en particulier à l'équation des extrémales d'une intégrale double. En outre l'auteur étudie le cas où la courbe à l'infini de (1) dans l'espace (r, s, t) est précisément $rt-s^2=0$; dans ce cas, c'est $r^2+s^2+t^2$ qu'il faut essayer de majorer a priori; l'auteur prouve que c'est impossible dans certains cas, et que dans d'autres c'est possible et le problème est "bien posé."

G. Giraud (Bonny-sur-Loire).

Vernotte, Pierre. Méthode très générale pour étudier le début des perturbations régies par les équations aux dérivées partielles de la physique mathématique. Application à la chaleur et à l'hydrodynamique. C. R. Acad. Sci. Paris **210**, 42–44 (1940). [MF 1246]

In a former note [C. R. Acad. Sci. Paris **208**, 1712 (1939)] the author had set forth a method of solving partial differential equations of mathematical physics with the independent variables x (space coordinate) and z (time) by expansions of the form $\sum x^{rq}f_q$, where the f_q are functions of $u=z/x^p$. In the present note the author applies this method especially to the study of disturbances during the first moments.

E. Rothe (Oskaloosa, Iowa).

Rice, S. O. The electric field produced by a point-charge located outside a dielectric wedge. Philos. Mag. **29**, 36–46 (1940). [MF 1158]

The author first expresses the potential of a point-charge q at a point $(\rho_0, \theta_0, 0)$ in cylindrical coordinates in the form

$$V_0=\frac{iq}{\pi(\rho\rho_0)^{\frac{1}{2}}}P\int_{-\infty i}^{\infty i}Q_{\lambda-\frac{1}{2}}(s)\frac{\cos\lambda[\pi-|\theta-\theta_0|]}{\sin\lambda\pi}d\lambda,$$

where $2\rho\rho_0 s=\rho^2+\rho_0^2+z^2$. When a dielectric wedge with edge along the axis of x is placed into this field, the potential becomes V_a in air, V_w in the wedge, where

$$V_a=V_0+\frac{iq}{\pi(\rho\rho_0)^{\frac{1}{2}}}P\int_{-\infty i}^{\infty i}Q_{\lambda-\frac{1}{2}}(s)[a(\lambda)\cos\lambda\theta+b(\lambda)\sin\lambda\theta]d\lambda,$$

$$V_w=\frac{iq}{\pi(\rho\rho_0)^{\frac{1}{2}}}P\int_{-\infty i}^{\infty i}Q_{\lambda-\frac{1}{2}}(s)[c(\lambda)\cos\lambda\phi+d(\lambda)\sin\lambda\phi]d\lambda,$$

and $\phi=\pi-\theta$. The unknown functions $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, $d(\lambda)$ are determined by the boundary conditions on the surface of the wedge. The potentials V_a and V_w obtained in this way are then shown to satisfy Laplace's equation and the boundary conditions at infinity. Expressions for V_a and V_w are deduced by the calculus of residues. The corresponding formulae for a line charge parallel to the edge of the wedge are deduced by integration.

E. T. Copson (Dundee).

Wagner, Karl Willy. Über Begründung und Sinn der Operatorenrechnung nach Heaviside. Z. Tech. Phys. **20**, 301–313 (1939). [MF 1050]

This paper contains an exposition of the Heaviside operational calculus. The latter is introduced as usual by means of the first order differential equation $E=U+RC(dU/dt)$ by replacing d/dt by p, treating the latter as an algebraic quantity, solving for u, and interpreting the result by expanding in positive or negative powers of p and replacing the latter, respectively, by successive derivatives or integrals. A general foundation for the Heaviside calculus is established by means of the contour integral interpretation [due to the author, G. Giorgi and T. J. A. Bromwich]

$$f(p)1=A(t)=\frac{1}{2\pi i}\int\frac{f(p)}{p}e^{pt}dp,$$

where 1 is 1 for $t>0$ and 0 for $t<0$, while the integration is carried out in the complex p plane along a line parallel to the imaginary axis but to the right of any singular points possessed by the integrand. This interpretation is illustrated for the case of irrational functions of p which occur in applying the Heaviside method to the heat conduction equation or in connection with problems involving inductionless cables. The relation between the Bromwich integral and the Fourier integral is given by means of a Laplace integral

$$\frac{f(p)}{p}=p\int_0^\infty e^{-pt}A(t)dt$$

(introduced into operational calculus by Carson). Rules for treating arbitrary initial boundary conditions are given as well as for methods of obtaining asymptotic expansions.
H. Poritsky (Schenectady, N. Y.).

Fujiwara, Matsusaburo. Asymptotic expansions in the Heaviside's operational calculus. Proc. Imp. Acad., Tokyo **15**, 283–287 (1939). [MF 1134]

Two theorems are obtained giving conditions of validity of Heaviside's asymptotic expansion formula, a formula which is more clearly interpreted as an expansion of the inverse Laplace transform. A note by the author added in proof points out that his first theorem is a special case of one by Bourgin and Duffin [Amer. J. Math. **59**, 489–505 (1937). Theorem 1]. The final conditions [p. 287] in the first theorem are clearly needed in the second, although it is not so stated. *R. V. Churchill* (Ann Arbor, Mich.).

Bourgin, D. G. and Duffin, R. The Laplace Heaviside method for boundary value problems. Bull. Amer. Math. Soc. **45**, 859–869 (1939). [MF 771]

This paper deals with the solution of the linear differential system

$$\sum_{j=0}^{n} (a_j+xb_j)d^jy(x)/dx^j=r(x),$$

$$U_i(a, b)=\sum_{j=0}^{n-1} \alpha_{ij}y^j(a)+\beta_{ij}y^j(b)=d_1, \quad i=1, \cdots, n,$$

where a_j, b_j, α_{ij}, β_{ij} are constants. The solution is represented by means of the Mellin-Laplace integrals

$$y(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xp}F(p)dp, \quad F(p)=\int_a^b y(x)e^{-xp}dx.$$

The "operational image" $F(p)$ of $y(x)$ is given by

$$F(p)=\int_\infty^p G(p)[R(p)]dp/G(p)L_1(p),$$

where

$$L_0(p)=\sum_{j=0}^{n} a_jp^j, \quad L_1(p)=\sum_{j=0}^{n} b_jp^j,$$

$$G(p)=\exp\left[-\int^p [L_0(p)/L_1(p)]dp\right],$$

$$R(p)=e^{-xp}\sum_{j=0}^{n-1} y^j(x)\left\{L_0^{j+1}(p)+\left(x-\frac{d}{dp}\right)L_1^{j+1}(p)\right\}_a^b$$

$$-\int_a^b e^{-xp}r(x)dx,$$

after the boundary conditions in y have been attended to; $L_a^j(p)$ denotes the quotient resulting from the division of $L_a(p)$ by p^j.

An application is made to solutions of the partial differential equation

$$(a+x)y_{xx}+by_{tt}=r(x, t)$$

over a rectangular (x, t) domain. *H. Poritsky.*

Jaeger, J. C. The Laplace transformation method in elementary circuit theory. Math. Gaz. **24**, 42–50 (1940). [MF 1458]

The author points out the simplicity of the use of the Laplace transformation in solving the problems for which the operational calculus was designed. He demonstrates that only the integral calculus is needed to obtain those properties of the transformation which are sufficient for solving the elementary electric circuit problems. These properties are of course some of the basic rules of the operational calculus. The illustrations consist of several circuit problems, which are simple problems in ordinary differential equations; these are solved entirely by the material developed in a few paragraphs at the beginning of the paper.
R. V. Churchill (Ann Arbor, Mich.).

Amerio, Luigi. Il metodo della trasformazione di Laplace in un problema di propagazione dell'elettricità. Ist. Lombardo, Rend. **72**, 485–506 (1939). [MF 939]

The author first considers the well-known equations, for harmonic oscillations, of a finite and infinite chain of equal electric quadrupoles. Then he assumes the output side of the Nth quadrupole to be short circuited and obtains the corresponding equations for the potential and current in any quadrupole of order $k<N$; a simplification is obtained if the quadrupoles are taken symmetrical. Then the problem is generalized by considering a Pupin cable consisting of sections with homogeneously distributed line constants, each section being terminated with a self inductance. Thus a known expression due to Campbell is obtained. In the next sections the same problem is treated with the aid of the Heaviside operational method (Laplace transform) yielding the solution for the potential and current in any quadrupole for an excitation being an arbitrary function of the time, use being made of the Borel-Volterra "composition product." A further investigation of the eigenfunctions of the problem leads to the generalized Laguerre polynomials defined by

$$t^\alpha L_\mu^{(\alpha)}(t)=\frac{e^t}{\mu!}\left(\frac{d}{dt}\right)^\mu(t^\alpha e^{-t}),$$

so that the solution can be written in the form of an infinite series of these polynomials. *B. van der Pol* (Eindhoven).

Droste, H. W. Ein Satz der Laplaceschen Funktionentransformation über die Aufteilung in Dauer- und Ausgleichsvorgang bei Gleich- und Wechselstrom und der Ausgleichssatz der komplexen Umwandlung. Elektr. Nachr. Techn. **16**, 253–257 (1939). [MF 1183]

This paper contains a treatment of some of the more familiar phases of operational calculus, based on the Laplace (or Carson) integral. The results are illustrated on the ordinary differential equation $(R+L\partial/\partial t)I(t)=U(t)$ and on the transmission line equations $(R+L\partial/\partial t)I=-\partial U/\partial x$, $(G+C\partial/\partial t)U=-\partial I/\partial x$. *H. Poritsky.*

Picone, M. Nuovi metodi d'indagine per la teoria delle equazioni lineari a derivate parziali. Rend. Sem. Mat. Fis. Milano **13**, 25 pp. (1939). [MF 1298]

The Laplace transform on a finite interval is used to show that certain boundary problems in partial differential equations of the type $a_{20}v_{xx}+2a_{11}v_{xt}+a_{02}v_{tt}+a_{10}v_x+a_{01}v_t+a_{00}v=f(x, t)$, $v(x_1, t)=v_1(t)$, $v(x_2, t)=v_2(t)$, $v(x, 0)=\varphi_0(x)$, $v_t(x, 0)=\varphi_1(x)$, can have not more than one solution of a sufficiently regular character in a region $x_1\leqq x\leqq x_2$, $0\leqq t\leqq T(x)$. The a's are continuous functions of x alone; and the boundary conditions are modified in an appropriate manner for some selections of those coefficients, or when x_2 is infinite. An undetermined positive function $T(x)$ is introduced in writing the following transform of the difference $u(x, t)$ of any two solutions: $u^*(x, \lambda)=e^{\lambda T}\int_0^T e^{-\lambda t}udt$. The transform

u^* satisfies a corresponding ordinary differential equation in x, and vanishes at x_1 and x_2. The function $T(x)$ can be determined for several moderate specializations of the a's, including fairly general hyperbolic and parabolic cases of the original equation, so as to find a suitable majorizing property of u^* for $\lambda \rightarrow +\infty$. The author then applies a theorem from an earlier paper [Atti Accad. Naz. Lincei. Rend. (6a) **28** (1938)] showing that, if $u^*(x, \lambda) < K\lambda^p$ as $\lambda \rightarrow +\infty$, for some real bounded functions $K(x)$ and $p(x)$, then $u(x, t)$ vanishes throughout the region $(x_1 \leqq x \leqq x_2,\ 0 \leqq t \leqq T(x))$, which depends on the function T. The method does not show the existence of a solution. *R. V. Churchill.*

Mangeron, D. L'applicazione del metodo di Picone, della trasformata di Laplace ad intervallo d'integrazione finito, alla teoria delle equazioni a derivate parziali d'ordine qualunque. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 1–9 (1939). [MF 1299]

The author applies Picone's method [Rend. Sem. Mat. Fis. Milano **13** (1939)] to show the uniqueness of the solutions of certain classes of boundary problems in linear partial differential equations of order n with two independent variables x and t. The unknown function and an appropriate set of its normal derivatives are prescribed on the lines $x=x_1$, $x=x_2$, $t=0$ within the region bounded by these lines and a curve $t=T(x)$, where $T(x)$ is a positive function in (x_1, x_2). Conditions in the form of inequalities are imposed upon the coefficients of the differential equation, which are functions of x alone, under which a function $T(x)$ can be determined such that not more than one solution can exist in the above region. Several cases of fourth order equations are discussed. The existence of solutions is not established. *R. V. Churchill* (Ann Arbor, Mich.).

Tranter, C. J. Note on a problem in the conduction of heat. Philos. Mag. **28**, 579–583 (1939). [MF 1114]

Contour integrals are used to solve two problems of temperatures in composite infinite solids. It would have been simpler and more direct to use the Laplace integral transformation. The solution of the first problem can be written at once by integrating the known Green's function for that problem [Carslaw, The Conduction of Heat, 1921, p. 163]. The formula given here [p. 583] for the temperature $v_2(x, t)$ in the second problem is incorrect; the required conditions at the planes of separation of the media are not satisfied. *R. V. Churchill* (Ann Arbor, Mich.).

Lowan, Arnold N. On some problems in the diffraction of heat. Philos. Mag. **29**, 93–99 (1940). [MF 1161]

Laplace transforms are used to obtain formulas for the temperatures in a semi-infinite solid whose boundary plane is insulated except for an aperture in the insulation. Through this aperture there is a prescribed flux of heat into the solid. Four cases are treated, for apertures of various shapes. The solutions are all formal in the sense that no conditions on the arbitrarily prescribed flux-functions are determined which are sufficient to show that the result satisfies the conditions of the boundary value problem. Some of the results are simple enough that it should not be difficult to establish them rigorously. *R. V. Churchill.*

Functional Equations

Kesava Menon, P. Some properties of binomial coefficients. Math. Student **7**, 93–96 (1939). [MF 1443]

The author obtains identities between factorials and binomial coefficients by using the polynomial operators of the calculus of finite differences. *M. Ward.*

Lancaster, Otis E. Some results concerning the behavior at infinity of real continuous solutions of algebraic difference equations. Bull. Amer. Math. Soc. **46**, 169–177 (1940). [MF 1266]

The problem under consideration refers, for real x large, to properties of real continuous solutions of algebraic difference equations of order m, (1) $P=0$, where P is a polynomial with real coefficients in $x, y(x), y(x+1), \cdots, y(x+m)$. In P a certain term T' is called the principal term. The proofs are largely based on the limits, as $x\rightarrow\infty$, of the ratios of T' to the other terms. With the notation of Hardy $(e_1(v)=\exp v,\ e_n(v)=\exp e_{n-1}(v),\ l_1(v)=\log v,\ l_n(v)=\log l_{n-1}(v))$ the typical results may be stated as follows. A solution of (1) (with $m=1$) cannot equal or exceed $v_n(x)=Ce_2(xl_n(x))$ for all $x>x_0(n)$ (constant $C>0$; any integer n); $v_n(x)$ $(C\neq 0$; integer $n)$ cannot be a solution of (1). If $y(x)$ is a solution of (1) and if for some α the sequence $u_n(n=0, 1, \cdots)$, with $u_n=y(\alpha+n)^k/y(\alpha+n+1)$ (any rational k), is monotone, then $y(x)$ does not exceed $v_n(x)$, for any integer n, for all $x>x_0(n)$. Given a function $\phi(x)$ with an arbitrary rate of increase, there exists an equation (1) (with $m=2$), having a solution $f(x)$, where $|f|>\phi$ for $x=x_j$ $(\lim_j x_j=\infty)$.

The above developments are analogous to certain results previously obtained by Borel, Vijayaraghavan and a number of others in the field of algebraic differential equations. *W. J. Trjitzinsky* (Urbana, Ill.).

Gross, George Lloyd. Use of functionals in obtaining approximate solutions of linear operational equations. Iowa State Coll. J. Sci. **14**, 37–38 (1939). [MF 743]

Abstract of a thesis.

Nardini, Renato. Sulla risoluzione di due equazioni funzionali del tipo di Volterra. Boll. Un. Mat. Ital. (2) **2**, 25–32 (1939). [MF 1308]

This paper is a survey of the results of Nardini's thesis [Bologna, 1939]. He treats some functional equations of Volterra type in a similar way as did L. Tonelli [Bull. Calcutta Math. Soc. **20**, 31 (1928)] and D. Graffi [Ann. Mat. Pura Appl. (4) **9**, 143 (1931)]. The typical equation is (1) $\phi(x)=f(x)+A[x, \phi(y)]$, $|y|\leqq|x|$, $-1\leqq x\leqq 1$, where $f(x)$ is a given continuous function and A a given functional of the unknown function $\phi(y)$ in $(-x, x)$. Their procedure corresponds to Cauchy's method of approximating solutions of differential equations by polygons, rather than to the method of successive approximation which is generally used for equations of Volterra type. Here Nardini uses a sequence defined by $\phi_n(x)=f(x)$ if $|x|\leqq 1/n$, $\phi_n(x)=f(x)+A[x\mp 1/n, \phi(y)]$ if $x>1/n$, $<-1/n$, respectively, where $|y|\leqq|x|-1/n$. He states conditions under which he can prove that $\lim \phi_n(x)$ exists and solves (1); they generalize the well-known conditions for differential equations and, if he adds a condition of Lipschitz type, he finds the uniqueness of the solution too. This problem includes as special cases Volterra's integral equations. Finally, he treats an analogous problem for functions of several variables. *E. D. Hellinger.*

Ganapathy Iyer, V. On certain functional equations. J. Indian Math. Soc. **3**, 312–315 (1939). [MF 1578]

The author proves that if $P(z)$ and $Q(z)$ are integral functions and if $P^2+Q^2=1$, then there exists an integral function $f(z)$ such that $P=\cos[f(z)]$, $Q=\sin[f(z)]$. By

means of the Picard theorem it is also shown that there exist no integral functions $P(z)$ and $Q(z)$ satisfying $P^n+Q^n=1$ for any integer $n\geqq 3$. There are other related results.

N. Levinson (Cambridge, Mass.).

Vescan, T. The solution of a functional equation dependent on a supplementary condition. Bol. Mat. **12**, 260–264 (1939). (Spanish) [MF 899]

The author verifies that the functional equation

$$f\left(\frac{1}{x_1x_2}\right)=f\left(1+\frac{x_2}{x_1}\right)+f\left(1+\frac{x_1}{x_2}\right),$$

with the condition $x_1+x_2=1$, is satisfied both by the linear function and by the logarithmic function. Two other functional equations, similarly dependent on a supplementary condition, are proposed as a theme of study.

A. González Domínguez (Providence, R. I.).

Jacobsthal, Ernst. Über die Funktionalgleichung $f(x+y)=f(x)+f(y)$. Norske Vid. Selsk. Forh. **12**, 74–75 (1939). [MF 1027]

The well-known fact that, if $f(x+y)=f(x)+f(y)$ and if $f(x)$ is bounded in $(0, \epsilon)$, then $f(x)=ax$, is proved by considering $\phi(x)=-x+f(x_0x)/f(x_0)$, where $f(x_0)\neq 0$. It is readily seen that $\phi(x)$ has any rational number as period, hence is bounded. Hence, if $M=\text{l.u.b.}\ |\phi(x)|$, we have $|\phi(2x)|=2|\phi(x)|\leqq M$, or $M=0$. *W. Feller* (Providence, R. I.).

MECHANICS

García, Godofredo. General solution of the ballistic problem, taking into account the corrections for the sphericity of the planet, the density of the medium, gravity and the effect of the wind. Revista Ci., Lima **41**, 309–337 (1939) = Actas Acad. Ci. Lima **2**, 107–135 (1939). (1 plate) (Spanish) [MF 1705]

García, Godofredo. The influence of the rotation of the planet on the movement of a projectile in a resisting medium. Revista Ci., Lima **41**, 339–348 (1939) = Actas Acad. Ci. Lima **2**, 137–146 (1939). (1 plate) (Spanish) [MF 1706]

Larrea Bancayan, Manuel. Motion of a projectile in a resisting medium. Some frequently used applications taking into account the resistance of the medium. Revista Ci., Lima **41**, 691–703 (1939). (Spanish) [MF 1720]

Cattaneo, Carlo. Libera caduta di un solido pesante con riguardo alla rotazione terrestre. Boll. Un. Mat. Ital. (2) **1**, 445–451 (1939). [MF 1315]

Somigliana, Carlo. Complementi alla teoria del campo gravitazionale ellissoidico. Ist. Lombardo, Rend. **72**, 91–101 (1939). [MF 945]

Vengono proposte nuove formole per le espressioni della gravità alle estremità degli assi del geoide ellissoidico e se ne indicano varie applicazioni e verifiche.

Author's summary.

Rosenblatt, Alfred. Sur les fondements mathématiques de la méthode de résistivité dans la prospection électrique du sous-sol. Revista Ci., Lima **41**, 475–484 (1939) = Actas Acad. Ci. Lima **2**, 145–154 (1939). [MF 1798]

Dynamics, Celestial Mechanics, Relativity

Sestini, G. Sulla dinamica di un particolare sistema piano. Boll. Un. Mat. Ital. (2) **1**, 436–444 (1939). [MF 1314]

Consider a vehicle idealized as follows: Two rigid bodies S_0 and S, free to move in a horizontal plane, are hinged to each other at a point P_0. Let A be a point rigidly fixed in S_0 and P a point fixed in S, both A and P being in the same horizontal plane with P_0. The system is constrained to move so that AP_0 is always tangent to the path of P_0 and PP_0 is tangent to the path of P. The reactions due to these constraints, which are evidently of non-holonomic type, are studied in detail. *D. C. Lewis* (Durham, N. H.).

Castoldi, Luigi. Osservazioni sulla funzione hamiltoniana e sull'energia totale di un sistema dinamico. Boll. Un. Mat. Ital. (2) **1**, 451–457 (1939). [MF 1316]

In a holonomic conservative dynamical system in which the constraints are not necessarily independent of the time, the Hamiltonian function H is obviously not necessarily equal to the total energy E. In this note it is shown that a necessary and sufficient condition that $E=H$ during an arbitrary motion is that the constraints be independent of the time. The further discussion for the case in which $E-H$ is a constant of the motion appears unsatisfactory. For instance, a motion of the system is completely determined by the values of the Lagrangian coordinates and their time derivatives at a single instant; and to preassign these values at two instants, as the author does, is to impose too many conditions. *D. C. Lewis* (Durham, N. H.).

Pastori, Maria. Propagazione delle azioni gravitazionali ed elettromagnetiche. Ist. Lombardo, Rend. **72**, 409–417 (1939). [MF 935]

By considering characteristics the author shows that Newtonian gravitational discontinuities are propagated instantaneously, and relativistic gravitational and electromagnetic discontinuities with the velocity of light. The only claim to originality is in the establishment of the last result without assuming the vanishing of the divergence of the 4-potential. *J. L. Synge* (Toronto, Ont.).

Caldirola, P. Su alcune relazioni fra le proprietà geometriche di una V_n e la dinamica delle particelle. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 19–23 (1939). [MF 1335]

The principal property discussed is formulated as follows: Given an arbitrary geodesic g in an n-dimensional Riemannian space V_n, it is always possible to find a V_{n+1} such that g is the projection on V_n of a nul-geodesic in V_{n+1}. Following Proca, Goudsmit [cf. C. R. Acad. Sci. Paris **208**, 884–887 (1939)], the author suggests the imbedment of the usual V_4 of relativity theory in a V_6. The motion of a particle is to be given as a nul-geodesic in V_6 and the momentum components corresponding to the two additional dimensions are interpreted as electric charge and spin, at least in the case of Euclidean spaces. *D. C. Lewis.*

Fock, V. A. Sur le mouvement des masses finies d'après la théorie de gravitation einsteinienne. Acad. Sci. U.S.S.R. J. Phys. 1, 81–116 (1939). [MF 990]

The purpose of the paper is to obtain, without any assumption of geodesic character, the Newtonian equations of motion as a first approximation in the relativistic problem of n bodies, and to push the calculation for the field and for the energy-tensor in the bodies to a second approximation. The harmonic coordinates of Lanczos are used. No general technique of successive approximations is developed, and in the approximations used the basic ideas are not wholly clear. The author does not quote the work of Dröste [Nederl. Akad. Wetensch., Proc. **19**, 447 (1916)] or later writers on the n-body problem, nor the fact that the geodesic principle as a first approximation is a simple consequence of the vanishing of the divergence of the energy-tensor [A. S. Eddington, Mathematical Theory of Relativity, Cambridge, 1924, 126]. The treatment of the energy-tensor is not quite convincing. The author asserts the continuity of components of the energy-tensor corresponding to stress; we know, however, from the theory of elasticity that there may be discontinuity in stress-components at the surface of a body. Moreover, in the expressions given for the energy-tensor in the second approximation there is no reference to the elastic properties of the bodies, analogous to the assumptions of fluidity and incompressibility which are essential features of the Schwarzschild interior solution. The author remarks that it was only when his paper was in press that he saw the somewhat similar paper by Einstein, Infeld and Hoffmann [Ann. of Math. **39**, 65 (1938)]; he compares the two papers and concludes that the statements of the problem and the method of calculation are different. The outstanding difference lies in the fact that the energy-tensor of matter, extended though of small volume, plays a fundamental part in the present paper, whereas in the other paper matter was regarded as located on singular lines in space-time, and the energy-tensor did not appear at all. *J. L. Synge.*

Rosen, N. General relativity and flat space. I. Phys. Rev. **57**, 147–150 (1940). [MF 1029]

It is proposed that it is possible to map any Riemannian space with metrical tensor $g_{\mu\nu}$ onto a flat space (metrical tensor $\gamma_{\mu\nu}$) so that the coordinates of corresponding points are the same. Covariant differentiation, denoted by a comma, with respect to the $\gamma_{\mu\nu}$ is introduced. It is possible to impose covariant conditions on the metric in order that the $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ should be related. The relations selected are $g^{\alpha\beta}g_{\alpha\mu,\beta}-\kappa_{,\mu}/\kappa=0$, where κ^2 is the ratio of the determinant of the $g_{\mu\nu}$ to that of the $\gamma_{\mu\nu}$. In this way it is possible to obtain the coordinates in the Riemannian space which correspond to preassigned coordinates in the associated flat space.

G. C. McVittie (Eastbourne).

Rosen, N. General relativity and flat space. II. Phys. Rev. **57**, 150–153 (1940). [MF 1030]

The flat space associated with a Riemannian space [see the above review] is regarded as of "physical" significance but the path of a moving particle is a geodesic of the Riemannian space. The form of the geodesic in the flat space is worked out and gives motion under a Newtonian gravitational force, the particle tracing out the geodesic having variable mass. Some general remarks on the significance of the new theory are made.

G. C. McVittie (Eastbourne).

Rosen, N. Note on ether-drift experiments. Phys. Rev. **57**, 154–155 (1940). [MF 1031]

If the Riemannian space-time due to a constant weak gravitational field φ is regarded as a flat-space containing a medium whose index of refraction is $1-2\varphi$, it is possible to find the motion of the observer relative to the medium by optical experiments. It is suggested that this idea accounts for D. C. Miller's "ether-drift" experiments. The field φ is due to the distribution of matter in the universe as a whole. *G. C. McVittie* (Eastbourne).

Lees, A. The electron in classical general relativity theory. Philos. Mag. **28**, 385–395 (1939). [MF 1115]

The author considers an electron as a region of space-time enclosed by a finite boundary, space-like sections of which are closed surfaces. Throughout space-time, except on the boundary, the metric tensor $g_{\mu\nu}$ is to be determined from the Einstein field equations in which the stress energy tensor is given as that due to an electromagnetic field described by a vector potential κ_μ, which in turn satisfies the Maxwell equations for free space everywhere except on the boundary. The quantities $g_{\mu\nu}$ and κ_μ are required to be continuous everywhere including the boundary but may have discontinuous normal derivatives on the boundary. On the boundary $g_{\mu\nu}$ and κ_μ are assumed to satisfy the following conditions: (1) The Gaussian curvature of the boundary is constant. (2) The electric field is normal to and the magnetic field is tangential to the boundary, and

$$(3)\qquad [G_l^m-\tfrac{1}{2}g_l^mG]=\xi_{\underline{n}}{}^{n}g_l^m-\xi_{\underline{l}}{}^{\underline{m}}-\xi G_l^m,\quad m,l=2,3,4,$$

where G_l^m are some of the components of the Ricci tensor, the coordinate system is such that

$$g_{\mu\nu}dx^\mu dx^\nu=-k^2(dx')^2+g_{lm}dx^ldx^m,$$

in which the boundary is $x'=x_0'$ (a constant), ξ is an arbitrary scalar function defined over the boundary,

$$[U]=k\int_{x_0'-0}^{x_0'+0}U\,dx',$$

and $\lambda_{\underline{m}}$ denotes the covariant derivative of λ with respect to the "boundary tensor" g_{lm}.

The discussion which leads to the condition (3) does not seem adequate since the author does not take account of the fact that the components of the Ricci tensor with latin indices (that is, $G_m{}^l$) differ from the components of the Ricci tensor formed from the "boundary tensor" g_{lm}. The relations between conditions (1) to (3) and a variational problem are stated. The spherically symmetric static solution of the Einstein field equations for a sphere of charge ϵ and mass m are examined in light of these conditions. For this case it is stated that $\xi=m$. The author does not give a general interpretation of the function ξ which is introduced initially to circumvent condition (1).

A. H. Taub (Seattle, Wash.).

Dramba, Constantin. Singularités imaginaires du problème isoscèle plan des trois corps. C. R. Acad. Sci. Paris **210**, 131–133 (1940). [MF 1290]

A special three body problem is that in which the motion is in a plane, two bodies being of equal mass m, the distance between them being $2z$, the third body being of mass M and being at a distance R from each of the others, and at a distance z' from their center of gravity. There is an imaginary binary collision when $R<0$ and $z<0$. The author obtains solutions of the differential equations which give

z, z' and R as series of positive integral powers of $(t-t_0)^{1/5}$, where R is an infinitesimal of order 2/5 with respect to $(t-t_0)$. His results are closely related to results obtained by Uno and by Chazy. *E. J. Moulton* (Evanston, Ill.).

March, A. und Foradori, E. Ganzzahligkeit in Raum und Zeit. II. Z. Phys. **114**, 653–666 (1939). [MF 1185]

This is the second part of a paper in which the author considers a coordinate system, rods and clocks, all composed of discrete particles. In this part the concepts of velocity and acceleration with respect to this discrete structure are analyzed. The usual objection is that such a discrete structure cannot be reconciled with relativity theory, because l_0, the smallest distance, is not invariant. The author denies this and claims that l_0 ought to be considered as an invariant. He concludes that two particles which do not coincide and are at rest in one coordinate system k' may coincide in k, if k and k' move with uniform velocity relative to one another. *L. Infeld* (Toronto, Ont.).

Takeno, Hyôitirô. Cosmology in terms of wave geometry. (V) Universe with born-type electromagnetism. J. Sci. Hirosima Univ. Ser. A **9**, 195–216 (1939). [MF 672]

The symmetrical quantity $\omega_{\bar{A}B}$ is uniquely determined by the equations $\omega\gamma_i=\bar{\gamma}_i\omega$; $\gamma_{(i}\gamma_{j)}=g_{ij}$. It is supposed that the vector $u^i=\bar{\psi}\omega\gamma^i\psi$ satisfies the equation

$$u^j\nabla_j\cdot u^i=2(M\overset{1}{F}{}_j{}^{\cdot i}+N\overset{2}{F}{}_j{}^{\cdot i})u^j+Qu^i$$

[cf. Twatsuki, Mimura and Sibata, J. Sci. Hirosima Univ. Ser. A **8**, 187 (1938)]. The corresponding equation for ψ is of the form $\nabla_i\psi=\sum_i\psi$. The conditions of integrability of this equation are satisfied by five different spherically symmetric line elements. *J. Haantjes* (Amsterdam).

Sulaiman, Shah. The astronomical consequences of relativistic two-body problem. Philos. Mag. **28**, 227–230 (1939). [MF 1123]

Levi-Civita has given a treatment of the relativistic two-body problem [Amer. J. Math. **59**, 325–334 (1937)] which leads to an absolute acceleration for the common center of gravity of the two bodies. It is now shown that Levi-Civita's equation of motion, when the mass of one of the bodies tends to zero, does not reduce to the equation of motion of a particle in the Schwarzschild field, as it should do. It is concluded that Levi-Civita's approximate treatment of the two-body problem cannot be correct. *G. C. McVittie.*

Ganguly, H. K. On the permanency of configurations of rotating fluids with spheroids as surfaces of discontinuity of density. Z. Astrophys. **19**, 136–153 (1939). [MF 1398]

A body, consisting of a uniform spheroidal core and a uniform spheroidal shell of lesser density, is supposed to rotate about an axis, points at different distances from the axis rotating at different rates. The angular velocity on either bounding surface will increase from the equator towards the poles [see Appell, Traité de mécanique rationelle, vol. 4, no. 2 for similar results], except when the shell is a confocal spheroid. In this case the core and the outer fluid rotate as if rigid, the angular velocity increasing inward. However, in the case of a spherical core, the core rotates slower than the outer shell. Conditions are given so that an equilibrium configuration with a spherical core be possible; for example, the density of the core must be less than five-thirds the mean density of the whole body. *B. Friedman* (Chicago, Ill.).

Moghe, D. N. On the stability of equilibrium of an isolated fluid sphere. Proc. Indian Acad. Sci., Sect. A. **10**, 399–406 (1939). [MF 913]

The author discusses the stability towards spatial and temporal disturbances of the possible equilibrium states of an isolated fluid sphere in general relativity. He shows that the static solutions of Einstein's field equations given by Tolman [Phys. Rev. **55**, 365–373 (1939)] and the non-static solutions developed by him [Proc. Indian Acad. Sci., Sect. A **10**, 407 (1939); these Rev. **1**, 125 (1940)] are in general only partially stable. In one case, however, the equilibrium will be stable if the pressure and density of the sphere are slowly varying functions of the time. Then by considering the possible series of configurations of the sphere as it is distorted, the author shows that equilibrium is still only partially stable. Stable equilibrium is ensured only when $\partial g_{44}/\partial r=0$ for any configuration of the sphere. *B. Friedman* (Chicago, Ill.).

García, Godofredo et Rosenblatt, Alfred. Sur la formule de Stokes dans la théorie de la gravité. Revista Ci., Lima **41**, 349–457 (1939). [MF 1640]

Dans un travail publié précédemment [Bull. Sci. Math. (2) **63** (1939)] nous avons étudié une surface d'équilibre voisine d'une surface d'équilibre connue. Nous avons réduit le problème de trouver l'écart entre ces deux surfaces connaissant les intensités de la gravité aux points correspondants des deux surfaces à un système de deux équations intégrodifférentielles. Nous nous proposons, maintenant, d'étudier ce système dans le cas le plus simple de la sphère comme surface initiale. *Author's summary.*

Hydrodynamics, Aerodynamics

Ertel, Hans. Über ein allgemeines Variationsprinzip der Hydrodynamik. Abh. Preuss. Akad. Wiss. 1939, no. 7, 9 pp. (1939). [MF 1087]

The usual variational principle for inviscid fluids applies only in the piezotropic case (density a function of pressure only). By a simple argument employing Lagrangian instead of the more usual Eulerian coordinates, the author establishes a variational principle without this restriction:

$$\delta\int\int\int\int L\,da\,db\,dc\,dt=0,$$

$$L=\rho_0\left\{\frac{1}{2}\left[\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\right]-\Phi(x,y,z)\right\}+\Theta p(a,b,c,t),$$

where a, b, c are initial coordinates, $\rho_0(a, b, c)$ the density at $t=0$, Φ the potential of the body-forces, and

$$\Theta=\partial(x,y,z)/\partial(a,b,c).$$

In applying the principle, x, y, z are to be varied as functions of a, b, c, t with suitable boundary conditions; $p(a, b, c, t)$ is not varied. The Euler-Lagrange equations lead to the three hydrodynamical equations for x, y, z, the function p being of course also involved. By the addition of some extra terms to L the variational principle is extended to the case where the motion is referred to rotating axes. The author hopes

that the principle may be useful in the application of methods of approximation for the solution of the nonlinear equations of physical hydrodynamics. *J. L. Synge.*

Jacob, Caius. Sur les mouvements lents des fluides parfaits compressibles. Portugal Math. 1, 209–257 (1939). [MF 745]

Application de la méthode approchée de Tchapliguine [Ann. Sci. Univ. Moscou **1904**, 1–121] à l'étude des jets gazeux issus d'ajutages de formes quelconques. Comme on le sait, cette méthode donne des résultats assez satisfaisants pourvu que les vitesses en jeu ne dépassent pas une certaine fraction de la vitesse de propagation du son. Pour obtenir une idée du degré d'approximation obtenu l'auteur considère le jet gazeux s'écoulant à travers un ajutage limité par deux parois plans formant un angle quelconque. Sous la supposition que la configuration présente un plan de symétrie l'auteur avait, dans un mémoire antérieur, réussi à donner la solution exacte de ce problème [Bull. Sci. École Polytéch. Timişoara **1937**, 47–59, 224–244]. La comparaison avec les résultats du présent mémoire montre que pour des vitesses d'écoulement allant jusqu'à 5/9 de la vitesse du son l'approximation est assex bonne. La possibilité d'appliquer la méthode approchée aux mouvements avec discontinuité des vitesses est examinée. *W. Prager* (Istanbul).

Tamada, Kō. On the flow of a compressible fluid past a sphere. Proc. Phys.-Math. Soc. Japan **21**, 743–752 (1939). [MF 1131]

It was shown by Raleigh [Philos. Mag. **32**, 1 (1916)] that the velocity potential for the subsonic flow of a compressible fluid past a sphere can be expressed as a power series in terms of Mach's number M (which is the ratio of the undisturbed velocity U, divided by the velocity of sound c_0 for the undisturbed flow). The equation in question is

$$\nabla^2\varphi=\tfrac{1}{2}M^2\left(\frac{\partial\varphi}{\partial r}\frac{\partial q^2}{\partial r}+\frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}\frac{\partial q^2}{\partial\theta}\right)\Big/(1-\tfrac{1}{2}(\gamma^2-1)M^2(q^2-1)),$$

where

$$q^2=\left(\frac{\partial\varphi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2,$$

and boundary conditions are prescribed for $r=r_0$ and $r=\infty$. Raleigh himself computed the first two terms of this series; the author finds the third term. He gives some graphs showing numerical differences between Raleigh's and his approximation. *E. Reissner* (Cambridge, Mass.).

Simmons, N. Free stream-line flow past a vortex. Quart. J. Math., Oxford Ser. **10**, 283–298 (1939). [MF 1039]

The author considers the two-dimensional flow around a vortex, the (ideal) fluid issuing between two parallel walls C_1B_1 and C_2B_2 and furthermore flowing between two lines B_1A_1, B_2A_2, which are supposed to be free stream lines, $A_k=C_k=\infty$, $k=1, 2$, $B_k=\pm(d+\frac{1}{2}i\pi)$. In the usual way the function dw/dz is introduced, w being the complex potential. The area of motion is mapped in an appropriate way into the upper half plane of the u-plane. The author shows that $dw/du=Q_2(u)(u-u_0)^{-1}(u-u_1)^{-1}(u-\bar{u}_1)^{-1}$, $Q_2(u)$ being a quadratic with real coefficients, u_0, u_1, $\bar{u}_1$ appropriate constants. The author studies the free stream lines, their asymptotes and furthermore expresses the physical data d, χ, z with the help of the constants u_k, $2\pi\chi$ being the vortex strength and z giving its location. At last a stream with free boundary only is discussed. *S. Bergmann.*

Simmons, N. Free stream-line flow past vortices and aerofoils. Quart. J. Math., Oxford Ser. **10**, 299–312 (1939). [MF 1040]

In the continuation of the paper reviewed above, the author discussed the following cases: (I) the vortex lies appreciably downstream, (II) the vortex lies near an edge of the orifice, (III) the vortex is weak. He gives approximate formulas for the calculation of the above mentioned constants from given physical data for these cases. Furthermore, he applies his results to the aerofoil theory; he considers, in particular, the case of an aerofoil placed symmetrically with respect to the wind tunnel and compares his results with those obtained by Toussaint with the help of the method of Prandtl. *S. Bergmann.*

Sakurai, Tokio. On the two-dimensional boundary layer equation for motion of viscous fluid near moving obstacle. Proc. Phys.-Math. Soc. Japan **21**, 707–712 (1939). [MF 1129]

The author considers the two-dimensional motion of viscous fluid produced by a moving obstacle, velocity of which varies with time. Starting from Navier-Stokes equations, the author gives the boundary layer equation expressed with the aid of coordinates which move with the obstacle. Let ϕ represent the stream function of the flow in the boundary layer and $u_x=\partial\phi/\partial y$, $u_y=-\partial\phi/\partial x$ the components of the velocity of fluid at arbitrary point $P(x, y)$ relative to the moving obstacle, then ϕ satisfies: $\partial u_x/\partial t+u_x\partial u_x/\partial x+u_y\partial u_x/\partial u_y=\partial\bar{u}/\partial t+\bar{u}\partial\bar{u}/\partial x+\nu\partial^2u_x/\partial y^2$, $\bar{u}$ being the limiting value of u_x at the end of the boundary layer and ν the coefficient of kinetic viscosity.
S. Bergmann (Cambridge, Mass.).

Sakurai, Tokio. New method of evaluating the flow in boundary layer which varies with time. Proc. Phys.-Math. Soc. Japan **21**, 632–637 (1939). [MF 1127]

In this note (continuation of the paper reviewed above) the author finds certain approximations for the stream for small t ($t=0$ being the moment when the obstacle begins to move). He considers the solution of $\partial u_x/\partial t-\partial\bar{u}/\partial t-\nu\partial^2u_x/\partial y^2=0$ for which $u_x(x, 0)=(\bar{u})_{t=0}$ as the first approximation. It can be expressed in the form of an integral. With the help of the first approximation he calculates then the second approximation. Analogously he deals with the case when flow in boundary layer changes from one state to another. *S. Bergmann* (Cambridge, Mass.).

Giraud, Georges. Sur un cas où un corps pesant tournant, consistant en un noyau solide entouré d'une masse liquide, est en équilibre relatif stable. C. R. Acad. Sci. Paris **209**, 620–623 (1939). [MF 706]

A solid body immersed in a uniform fluid is in rotation about an axis with constant angular velocity, the forces on the body and fluid being their mutual gravitational attraction. It is shown that this motion can be stable if the density of the body is greater than that of the liquid and if the free surface is a surface of revolution having u small, where $\sin u$ is the eccentricity of its meridian.
G. C. McVittie (London).

Giraud, Georges. Petits mouvements relatifs périodiques d'un corps pesant tournant, constitué par un noyau solide immergé dans une masse liquide homogène. C. R. Acad. Sci. Paris **209**, 661–663 (1939). [MF 709]

A solid body is supposed to be immersed in a perfect fluid and to rotate with angular velocity ω about an axis.

The elements of the fluid and of the body attract each other according to Newton's law. A force, variable with position and periodic with time, with period $2\pi/\lambda$ is applied to the system. It is required to find the vibrations of period $2\pi/\lambda$ of the system about the steady motion, the force being assumed to be small. The equations which would give the solution of the problem are written down but are not solved. If λ lies in $-2\omega \leqq \lambda \leqq 2\omega$, it is pointed out that integral equations would be needed for a complete solution. The possibility that λ might turn out to be real is indicated, in which case the motion would be unstable.

G. C. McVittie (London).

Dasgupta, Hiranya Kumar. Sur la stabilité de deux files de tourbillons dans un canal de largeur finie. C. R. Acad. Sci. Paris **209**, 503–505 (1939). [MF 524]

The author studies the stability of vortex filaments in a channel of width ω_1; the system is supposed to be moving with a constant velocity. The small displacements of the kth vortex in the κth filament are denoted by $\zeta_k^{(\kappa)}(t)$, $\kappa=1, 2$, and are supposed to be periodic, that is, $\zeta_{k+np}^{(\kappa)}=\zeta_k^{(\kappa)}$; n, p integers. The author establishes the differential equations for $\zeta_k^{(\kappa)}(t)$ in the usual way and shows that $\theta^{(1)}(t)-\theta^{(2)}(t)$ is always bounded and periodic, $\theta^{(\kappa)}(t)=\sum_{\nu=1}^{n}\zeta_\nu^{(\kappa)}(t)$. Furthermore, he indicates certain conditions that $\zeta_k^{(1)}(t)-\zeta_k^{(2)}(t)$ be bounded and periodic.

S. Bergmann.

***Burgers, J. M. Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion.** Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. 17, no. 2, 53 pp. (1939). f 1,75.

The author's aim is to discuss some simple systems of equations giving rise to statistical problems, like those of hydrodynamics and easier to study. The first example with a laminar and a turbulent solution is furnished by the two equations

$$(1) \qquad dU/dt=P-\nu U-v, \quad dv/dt=Uv-\nu v,$$

where P and ν are constants. The laminar solution $U=P/\nu$, $v=0$ is stable so long as $U<\nu$, or $P<\nu^2$. When $P>\nu^2$, stationary solutions $U=\nu$, $v=\pm(P-\nu^2)^{\frac{1}{2}}$ are stable for small disturbances. The rate of decay of a small disturbance is determined by the equation $\lambda^2+\lambda\nu+2(P-\nu^2)=0$. When $P=\nu^2$, one root is zero and so the stable character of the turbulent solution is just disappearing. In the second example there are three equations

$$(2) \qquad \begin{gathered} dU/dt=P-\nu U-v^2-w^2, \quad dv/dt=Uv-Uw-\nu v, \\ dw/dt=Uv+Uw-\nu w. \end{gathered}$$

The laminar solution is again $U=P/\nu$, $v=w=0$, but the turbulent solution $U=\nu$, $v=V\cos U(t-t_0)$, $w=V\sin U(t-t_0)$, $V=(P-\nu^2)^{\frac{1}{2}}$ is no longer independent of the time; it is again stable for small disturbances.

To obtain a system in which the turbulent motion has a spatial distribution the equations

$$(3) \qquad \begin{gathered} dU/dt=P-\nu U-\int_0^1 v^2 dy, \\ \partial v/\partial t=Uv+\nu\partial^2 v/\partial y^2-(\partial/\partial y)(v^2) \end{gathered}$$

are first discussed. The laminar motion $U=P/\nu$ is now stable if $P<\nu^2\pi^2$. A corresponding system with two secondary variables v, w is introduced to complete the discussion. The treatment of the turbulent solutions requires an examination of the properties of the integral

$$\int[C-s+\log(1+s)]^{-\frac{1}{2}}ds/(1+s).$$

This is done with the aid of inequalities. An equation in which the integral between the limits η_1 and η_2 is equal to $(2U/m^2\nu)^{\frac{1}{2}}$ fixes the value of C when U is given. Graphs are drawn to illustrate the cases $m=2$ and $m=3$. The relation between U and P is found for the turbulent solution $m=1$ and it is shown that for sufficiently great values of U there is a quadratic resistance law. A boundary layer theory is developed by supposing that ν is small and U/ν large. Solutions of (3) in which v varies with time are next considered and attention is directed to the life history of the coefficients of a set of elementary functions.

The last half of the work is now devoted to the case of two secondary variables v, w. In investigating the turbulent solutions there are boundary regions to be considered and also an interior domain situated between the boundary regions. In this domain the equations

$$\begin{gathered} \partial v/\partial t=U(v-w)-2v\partial v/\partial y+2w\partial w/\partial y, \\ \partial w/\partial t=U(v+w)+2w\partial v/\partial y+2v\partial w/\partial y \end{gathered}$$

are considered.

H. Bateman (Pasadena, Calif.).

Burgers, J. M. Application of a model system to illustrate some points of the statistical theory of free turbulence. Nederl. Akad. Wetensch., Proc. **43**, 2–12 (1940). [MF 1233]

The assumptions of the theory of uniform isotropic turbulence are applied to the model system

$$\begin{gathered} \partial v/\partial t=U(v-w)+\nu\partial^2 v/\partial y^2-2v\partial v/\partial y+2w\partial w/\partial y, \\ \partial w/\partial t=U(v+w)+\nu\partial^2 w/\partial y^2+2w\partial v/\partial y+2v\partial w/\partial y, \end{gathered}$$

the exposition of von Kármán and Howarth being taken as a guide. The equation for the propagation of correlation, obtained by a process of averaging, is

$$\frac{\partial}{\partial t}(f\overline{v^2})=2Uf\overline{v^2}+2\nu\overline{v^2}\partial^2 f/\partial r^2.$$

When the term in U is dropped, this is the analogue of an equation obtained by von Kármán and Howarth.

A direct investigation is made of the propagation of an elementary region of turbulence in the model systems. Attention is paid mainly to a model system with only one variable v, which (with $U=0$) is governed by the equation

$$\partial v/\partial t=\nu\partial^2 v/\partial y^2-2v\partial v/\partial y.$$

An estimate of the minimum thickness to which a vortex can be drawn out is based on the equation

$$r\partial\sigma/\partial t=Ur^2\partial\sigma/\partial r+\nu(r\partial^2\sigma/\partial r^2-\partial\sigma/\partial r).$$

The solution which, as $t\to\infty$, approaches the limiting form $\sigma=C\{1-\exp(-r^2U/2\nu)\}$ is regarded as appropriate.

H. Bateman (Pasadena, Calif.).

Ray, Manohar. On turbulent liquid motion outside a circular boundary. Philos. Mag. **28**, 231–240 (1939). [MF 1124]

Starting from G. I. Taylor's general equations of motion in a turbulent boundary layer in the form given by Goldstein [Proc. Cambridge Philos. Soc. **31** (1935)], the equations for two-dimensional motion outside an infinite cylinder are obtained in terms of x, the arc length of the section, and y, the distance along the normal. These equations are applied to a circular cylinder of radius r on the assumption that the region of turbulence is small. The solution is then obtained, by successive approximations, in the form $u=f_1(y)\sin(x/r)$, $v=f_2(y)\cos(x/r)$, $y=x/r$, and the functions $f_1(y)$, $f_2(y)$ are expressed in descending powers of $\log y$. Numerical calculations up to $y=0.3$, the limit of the validity

of the approximations, disclose a retrograde u-component in the turbulent layer near the wall, gradually changing into a direct one as y increases. The solution, having been adjusted to make $\partial u/\partial y$ infinite at the wall, cannot be expected to give any information regarding the viscous layer very near the boundary. *L. M. Milne-Thomson.*

Possio, Camillo. Sul moto non stazionario di una superficie portante. Atti Accad. Sci. Torino 74, 285–299 (1939). [MF 512]

Making use of Prandtl's acceleration potential the author investigates the aerodynamic field around a thin wing animated by an arbitrary motion during which, however, the angle of incidence remains small. The translational motion of a thin flat wing is treated as a particularly simple case, further examples being reserved for a separate publication.

W. Prager (Istanbul).

Bateman, H. The aerodynamics of reacting substances. Proc. Nat. Acad. Sci. U.S.A. **25**, 388–391 (1939). [MF 886]

A series of superposed fluids $S_1, \cdots, S_n$ are considered which may interact chemically. The density of S_r is ρ_r which is a function of position (x, y, z) and of time t. The motion of the fluids is irrotational, the velocity components (u_r, v_r, w_r) of S_r being derivable from φ_r. The changes taking place in the mixture are governed by a function f which is a function of x, y, z, t and of the densities and velocity-potentials. A general variational principle is considered, namely:

$$0=\delta\int d(x, y, z, t)\left[f-\sum_{r=1}^{n}\rho_r\left(\frac{\partial\varphi_r}{\partial t}+\tfrac{1}{2}u_r^2+\tfrac{1}{2}v_r^2+\tfrac{1}{2}w_r^2\right)\right],$$

and it is shown that the total pressure is

$$p=f-\sum_{r=1}^{n}\rho_r\frac{\partial f}{\partial\rho_r},$$

whilst the partial pressures can also be derived. The theory can be extended to rotational motion. If $\partial f/\partial\varphi_r$ is not zero, the equation of continuity for S_r is not satisfied which allows for this substance being created or destroyed by chemical action. *G. C. McVittie* (London).

✱Chapman, Sydney and Cowling, T. G. The Mathematical Theory of Non-uniform Gases. Cambridge University Press, Cambridge, 1939. xxiii+404 pp. $7.50.

In order to abbreviate the formulae in this difficult subject a vector notation is adopted and explained in the first chapter. The divergence of a vector quantity φ is denoted by $(\partial/\partial x)\varphi$ and, if φ is a function of a second vector C whose components are U, V, W, the symbol $(\partial/\partial C)\varphi(C)$ is used to denote the expression

$$\frac{\partial\varphi_x}{\partial U}+\frac{\partial\varphi_y}{\partial V}+\frac{\partial\varphi_z}{\partial W},$$

where $\varphi_x, \varphi_y, \varphi_z$ are the three components of the vector φ. The expression on the left hand side of Boltzmann's equation is also abbreviated into

$$\frac{\partial f}{\partial t}+C\frac{\partial f}{\partial r}+F\frac{\partial f}{\partial C},$$

or $\mathfrak{D}f$. The tensor notation is next described and different types of tensors are introduced, such as the velocity gradient tensor and the related tensors which give the rate of strain and the rate of shear. Chapter 2 deals with the properties of a gas and some basic definitions and theorems. The equations of Boltzmann and Maxwell are then developed, Enskog's generalization of Maxwell's equation of transfer being given. The theory of molecular encounters is then presented. The next three chapters are concerned with Boltzmann's H-theorem, the Maxwellian velocity distribution, the free path, persistence of velocities and the elementary theory of the transport phenomena.

In the important chapter on the non-uniform state for a simple gas, use is made of Enskog's method of solving the integral equation and of Burnett's calculation of certain quantities A and B with the aid of Sonine's polynomials. The non-uniform state for a mixture of gases is discussed in the next chapter. General expressions are first found and then formulae are obtained for special molecular models. The results of the analysis are of much physical interest and a large part of the book is devoted to the comparison of theory and experiment for viscosity, thermal conduction and diffusion. The appearance of the book is timely on account of the interest which is now being taken in thermal diffusion, a phenomenon which has been made known to physicists largely by the researches of Chapman. To the list of experimental papers on thermal diffusion on page 258 may now be added J. Bardeen [Phys. Rev. **57**, 35–41 (1940)] and A. O. Nier [Phys. Rev. **57**, 30–34 (1940)]. The third approximation to the velocity-distribution function is discussed in Chapter 15. Work of this kind was begun by Maxwell, carried further by Enskog and Lennard-Jones and completed by Burnett. The analysis is not given in full but is simplified so that the nature of certain quantities is ascertained. In the next chapter analysis for the discussion of dense gases is developed and an account is given of the unpublished work of H. H. Thorne on mixed dense gases. The quantum theory of molecular collisions and of transport phenomena forms the subject of Chapter 17. The wave-equation governing the relative motion in an encounter between two different molecules is solved by means of a series of Legendre functions with coefficients which are functions of the distance between the molecules. The last chapter deals with electromagnetic phenomena in ionized gases. An appendix gives Enskog's method of integration which differs from that used in the text (Chapter 9). The general Maxwell-Boltzmann distribution of velocities is also discussed. The book ends with a good historical account of the development of the kinetic theory of gases.

H. Bateman (Pasadena, Calif.).

Theory of Elasticity

Schuh, J. F. L'élasticité théorique, basée sur les principes énergétiques. Chr. Huygens **18**, 120–144 (1940). [MF 1562]

The paper is a discussion of the fundamentals of the theory of elasticity. With the principle of superposition as a base, the elastic energy is defined, Maxwell's reciprocity laws and Castigliano's formulae are derived as well as certain properties of the coefficients of the quadratic form representing the energy. *E. Reissner.*

Locatelli, P. Sopra il teorema del minimo lavoro per corpi non perfettamente elastici. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 1, 10–18 (1939). [MF 1334]

Starting from a general statement of the principle of virtual labor for continua, the author investigates the conditions under which for imperfectly elastical solids the minimum of labor theorem is valid. He shows that in addition to the type of elastoplastic solids as defined by Colonnetti [Comm. Pont. Accad. Sci. 2, no. 2: "La statica dei corpi elastoplastici"] a large class of solids can be defined for which this generalized Menabrea theorem is valid. As an example, the beam suspended in three points is discussed.

P. Nemenyi (Iowa City, Iowa).

Morris, Rosa M. The internal problems of two dimensional potential theory. Math. Ann. 117, 31–38 (1939). [MF 1381]

Let $T_i(T_e)$ be the transformation which maps the interior (exterior) of a curve C on a half plane. When C is a level curve $\Re(i\zeta)=0$ of one or more unclosed curves, T_e may be readily obtainable though T_i is not. In particular, when

$$(1)\qquad z=c[\cos \tfrac{1}{2}n(\zeta+i\alpha)-\cos \tfrac{1}{2}n\beta]^{2/n},$$

the level surface $R(i\zeta)=\alpha$ is an n-bladed cut of alternate long and short branches. The singular points of the transformation in this and the other examples are all on the interior equipotential.

The torsion problem is solved by a development of Wilton's method for the case when instead of (1) we have

$$z=e^{-i\zeta}\sum_{n=0}^{\infty} a_n e^{ni\zeta}.$$

Use is made of an expansion of the complex potential

$$\phi+i\psi=\tfrac{1}{2}iz(\zeta)\bar{z}(\zeta)+F(\zeta)$$

in a series of form $\sum_{n=-\infty}^{\infty} u_n e^{ni\zeta}$, $F(\zeta)$ being a "real" function. In the corresponding hydrodynamical problem

$$\phi+i\psi=wz(\zeta)-\tfrac{1}{2}i\omega z(\zeta)\bar{z}(\zeta)+F(\zeta),$$

and a brief discussion is given. *H. Bateman.*

Frola, E. L'estensione dei teoremi di Castigliano alla dinamica. Atti Accad. Sci. Torino 74, 438–447 (1939). [MF 1318]

Castigliano's principle, which expresses displacements as variations of the energy with respect to forces, is extended to the motion of an elastic body subjected to forces depending on the time. One of the author's theorems, for example, reads

$$\lim_{\sigma\to 0,\,\epsilon\to 0} \delta S/\rho\delta X(Q,\tau)V_\sigma\epsilon=u(Q,\tau);$$

u and X are displacement and force at the point Q and the time τ; σ is a neighborhood of Q with the volume V_σ; δS is the variation of Hamilton's integral extended over σ and a time-interval of the length ϵ. *K. O. Friedrichs.*

Arrighi, Gino. Note di elastodinamica. Ist. Lombardo, Rend. 72, 387–396 (1939). [MF 934]

If the vector $\mathbf{v}$ satisfies

$$\left(a^2\Delta-\frac{\partial^2}{\partial t^2}\right)\left(b^2\Delta-\frac{\partial^2}{\partial t^2}\right)\mathbf{v}=\mathbf{w}+\nabla f,$$

where f is a harmonic function and a and b are constants, then the vector

$$\mathbf{s}=a^2\nabla\nabla\cdot\mathbf{v}-b^2\nabla\times(\nabla\times\mathbf{v})-\frac{\partial^2\mathbf{v}}{\partial t^2}$$

satisfies

$$b^2\nabla\nabla\cdot\mathbf{s}-a^2\nabla\times(\nabla\times\mathbf{s})-\frac{\partial^2\mathbf{s}}{\partial t^2}=\mathbf{w}+\nabla\psi,$$

where ψ is again a harmonic function. Application of this result to the theory of elasticity. *W. Prager* (Istanbul).

Kappus, Robert. Zur Elastizitätstheorie endlicher Verschiebungen. II. Z. Angew. Math. Mech. 19, 344–361 (1939). [MF 1472]

The theory of Part I of this paper [Math. Rev. 1, 92 (1940)] is first applied to the torsion and flexure problems of a bar having large displacements but small strains. The principal application of the theory is made in a detailed investigation of the general problem of elastic stability both from the standpoint of the appropriate differential equations and from that of the energy method.

H. W. March (Madison, Wis.).

Pastori, Maria. Propagazione di un generico movimento in una membrana inestendibile. Ist. Lombardo, Rend. 72, 431–436 (1939). [MF 936]

The method of characteristics is used to discuss the propagation of waves of discontinuity in an inextensible membrane, the discontinuities being in the stress-gradient, acceleration and curvature. The conditions of dynamical compatibility lead to a determinantal equation of the sixth order for the propagation of the discontinuity. However this equation is satisfied identically, and the author, following a method of Levi-Civita, equates to zero the minor of highest order which is not identically zero, thus obtaining an expression for the velocity of propagation.

J. L. Synge (Toronto, Ont.).

Weatherburn, C. E. On transverse vibrations of curved membranes. Philos. Mag. 28, 632–634 (1939). [MF 1112]

The author shows that for a flexible membrane whose equilibrium state of tension is a minimal curved surface with mean curvature $J=0$, the transverse vibrations of the membrane having small displacements ξ and no normal loading, the equation

$$\rho\frac{\partial^2\xi}{\partial t^2}=T(\nabla^2\xi-2K\xi)$$

holds, where T is the uniform tension and K is the Gaussian curvature of the surface at the equilibrium position. For small vibrations of a normally loaded membrane the displacement is a solution of

$$\rho\frac{\partial^2\xi}{\partial t^2}=T[\nabla^2\xi-(2K-J^2)\xi]-\lambda J^2\xi,$$

where λ is a modulus number defining the elastic property of the membrane. *D. L. Holl* (Ames, Iowa).

Scholte, J. G. On the vibrations of an elastic sphere with central core. Nederl. Akad. Wetensch., Proc. 42, 918–929 (1939). [MF 977]

The purpose of the paper is to derive equations for the frequencies of vibrations of an elastic gravitating sphere. There are two kinds of such vibrations, those which involve no dilatation and no radial displacement and those for which the radial rotation component is zero. The equations obtained by the author are valid for a sphere consisting of a core and a surface layer of different elastic properties. For

the vibrations of the second kind which are more complicated, only a surface layer having the properties of an elastic fluid is considered. It is shown that if the wave length is small compared with the radius of the sphere the frequency equations become those for the corresponding surface waves in a half-space. The equations obtained are generalizations of results of Bromwich [Proc. London Math. Soc. **30** (1899)], Love [Some Problems of Geodynamics, Cambridge, 1911] and Sezawa [Bull. Earthquake Res. Inst. Tokyo (1937), (1938)]. No numerical results are given in the paper. *E. Reissner* (Cambridge, Mass.).

Iguchi, Shikazo. Die erzwungenen Schwingungen der allseitig eingespannten rechteckigen Platte. Proc. Phys.-Math. Soc. Japan **22**, 1–14 (1940). [MF 1471]

The author studies vibrations of a clamped elastic plate subjected to a periodically changing disturbing force. The deflection w of the middle plane of the plate is known to satisfy the equation

$$\nabla^2\nabla^2 w+\lambda^2\frac{\partial^2 w}{\partial t^2}=Af(x, y)\cos pt,$$

where λ and A are physical constants, and $f(x, y)\cos pt$ is the disturbing load. The method of solution consists of a formal superposition of free vibrations, characterized by the related homogeneous equation, onto the forced vibrations. The solution is assumed in the form of an infinite series of particular solutions with unknown coefficients. The calculation of coefficients is troublesome, as is illustrated by a discussion of the behavior of the plate loaded so that $f(x, y)=\text{const.}$ over a rectangular area whose boundary is parallel to the edges of the plate and is zero elsewhere. A monograph by A. Weinstein from the Mémorial des Sciences Mathématiques, fascicule no. 88 (1937), which bears on the problem discussed by the author, should be consulted. *I. S. Sokolnikoff* (Madison, Wis.).

Severina, Brusa. I coefficienti di elasticità nei corpi isotropi a strati e la loro determinazione. Ist. Lombardo, Rend. 72, 201–206 (1939). [MF 951]

The author considers an elastic material which is transversely isotropic with reference to the normals and tangent planes of a family of parallel surfaces. It is shown that the five coefficients of elasticity can be found experimentally by measuring the extensions and lateral contractions for extension in the direction of the axis of symmetry as well as for extension at right angles to this axis. *W. Prager.*

Morris, Rosa M. Some general solutions of St. Venant's flexure and torsion problem. I. Proc. London Math. Soc. (2) 46, 81–98 (1940). [MF 1452]

Formal solutions are given of the problems of torsion and flexure for a prism whose cross section is defined by the parametric equation

$$x+iy=\sum_{n=0}^{\infty} a_n e^{ni\xi}, \qquad 0<\xi<2\pi,$$

where the coefficients a_n are generally complex. The area S, the coordinates (x_G, y_G) of the centroid, the moments A, B and the product H of inertia are all expressed in terms of the coefficients a and a related quantity $b_r=\sum_{n=0}^{\infty} a_{n+r}\bar{a}_n$, where a bar over a symbol denotes the conjugate complex number. It can be assumed without loss of generality that $\sum_{n=1}^{\infty} na_n\bar{b}_n=0$. In the flexure problem use is made of complex potentials Ω_{12}, Ω_{21} defined by real functions $F_{12}(\zeta)$, $F_{21}(\zeta)$ such that

$$24\Omega_{12}=i[z(\zeta)+\bar{z}(\zeta)]^3+F_{12}(\zeta), \quad 24\Omega_{21}=F_{21}(\zeta)-[z(\zeta)-\bar{z}(\zeta)]^3,$$

and of a set of coefficients g_n which are certain bilinear functions of the a's and $\bar{b}$'s. In the torsion problem use is made of a complex potential

$$\Omega_3=\tfrac{1}{2}iz(\zeta)\bar{z}(\zeta)+F_3(\zeta)$$

and of a set of coefficients c_n which are bilinear functions of the quantities a_n, $\bar{a}_n$. A list is given of the sections to which the new method is applicable. *H. Bateman.*

Howland, R. C. J. and Knight, R. C. Stress functions for a plate containing groups of circular holes. Philos. Trans. Roy. Soc. London, Ser. A. **238**, 357–392 (1939). [MF 1271]

The paper is concerned with the determination of the biharmonic stress function for the following regions: (a) infinite plane regions which are exterior to a pair of circles, or to two pairs of circles, or to an infinite double row of circles; (b) that part of an infinite strip which is exterior to a pair of circles, or to two pairs of circles. In each of these cases the circles are of equal radii. The paper provides no general method of determining the stress function for a plate containing groups of circular holes, and depends on the extension of the special methods developed by the authors in several previous papers. *I. S. Sokolnikoff.*

Conforto, Fabio. Sopra un complemento all'equazione dei tre momenti per una trave continua inflessa e sollecitata assialmente. Ann. Mat. Pura Appl. **18**, 107–145 (1939). [MF 586]

An equation of three moments for a continuous beam subjected to axial and transverse forces is established under the supposition that in each span (1) the flexural rigidity of the beam varies in a linear manner, (2) the axial force is constant, (3) the transverse forces consist of: (a) a uniformly distributed load, (b) concentrated loads at arbitrary points of the beam. The coefficients of the equation of three moments thus obtained involve Bessel functions of the first order. Tables facilitating the use of the equation are in course of preparation. *W. Prager* (Istanbul).

Cornish, R. J. The magnitude of the direct stress in a beam of fixed span. Philos. Mag. **28**, 481–487 (1939). [MF 1117]

If the ends of a beam are held a fixed distance apart, a direct tension will be developed in the beam when lateral forces are applied to it. The author obtains solutions for the tension thus developed, assuming various types of lateral loads and end conditions. The tension, it appears, can always be given by a simple formula in terms of the maximum deflection and the span of the beam. In addition, the correct maximum deflection can be replaced in these formulas by the maximum deflection obtained by neglecting the direct tension without introducing appreciable errors.
J. J. Stoker (New York, N. Y.).

Grioli, G. Sollecitazioni di una struttura elicoidale incastrata agli estremi. Ann. Lavori Pubblici **1939**, 7 pp. (1939). [MF 1297]

La nota espone un procedimento per la determinazione delle sollecitazioni che si destano in una generica sezione normale di una trave elicoidale sulla quale agisce un carico uniformemente ripartito lungo l'elica assiale. Alla fine vengono dato i grafici corrispondenti ad un caso numerico trattato. *Author's summary.*

Dean, W. R. The distortion of a curved tube due to internal pressure. Philos. Mag. 28, 452–464 (1939). [MF 1116]

In a closed circular tube under constant internal hydrostatic pressure p, the displacement vector $\mathbf{d}=(u, v, w)$ satisfies the vector equation

$$(\lambda+2\mu)\ \text{grad div}\ \mathbf{d}-\text{curl curl}\ \mathbf{d}=0,$$

where (λ, μ) are Lamé's elastic constants. Due to axial symmetry, $w=0$ and (u, v) are functions of r, θ and R, the radial and latitude variable in a meridian section of the tube and the generating radius of the tube, respectively. By a method of successive approximations in which (u, v) are expanded in the forms

$$u=u_0+u_1/R+u_2/R^2+\cdots+\sin\theta\left(K_1R+\frac{K_3}{R^3}+\frac{K_5}{R^5}+\cdots\right)+\cdots,$$

$$v=v_0+v_1/R+v_2/R^2+\cdots+\cos\theta\left(K_1R+\frac{K_3}{R^3}+\frac{K_5}{R^5}+\cdots\right)+\cdots$$

with K as constants to be determined, sequential determination of displacements (u_0, v_0), (u_1, v_1), etc., and the corresponding stresses are made with the result that the first K becomes the average peripheral extension of the tube similar to the extension of a closed cylindrical tube under internal pressure. The stresses are compared with those stresses obtained by an approximation with thin shells with good agreement up to a certain order of a/t, where a is the radius of the circular section and t is its thickness, but this comparison is not possible for the displacements which show large deformations of the circular section in the axial direction. *D. L. Holl* (Ames, Iowa).

Frenkel, J. and Kontorova, T. On the theory of plastic deformation and twinning. Acad. Sci. U.S.S.R. J. Phys. 1, 137–149 (1939). [MF 991]

The authors study the behaviour of the following model which represents essential features of plastic deformation in crystals: an infinite chain of equidistant fixed atoms opposite a similar chain of atoms elastically bound to equidistant positions of equilibrium. Two types of wave-like motion in the latter chain are shown to be possible: (1) the propagation of small vibrations about the equilibrium positions; (2) the propagation of a slip wave, the equilibrium position of each atom being shifted to that of the next one. The slip wave travels with a velocity dependent on its energy, and approaching the velocity of sound as this energy increases. Slipping cannot occur unless this energy reaches a certain critical value. The preceding ideas are extended to the case of a plane layer of atoms and the results applied to the twinning of crystals. *W. Prager* (Istanbul).

MATHEMATICAL PHYSICS

***London, F. et Bauer, E. La théorie de l'observation en mécanique quantique.** Actual. Sci. Ind. 775. Exposés de physique générale. III. Hermann & Cie, Paris, 1939. 51 pp.

This treatise gives a concise exposition of the fundamental principles of quantum mechanics and their interpretation, avoiding the use of complicated mathematical methods, and the solution of specific proper value problems. The main mathematical device is the geometrical treatment of abstract Hilbert space. The statistical interpretation of quantum mechanics is discussed, the notions of pure and of mixed quantum mechanical assemblies are analyzed, and finally composite systems are studied in detail. This leads to an interesting exposition of the nature of correlations in quantum mechanics, and more specifically of the relations of the observer and the observed. A concrete and very instructive example of these general considerations is given by discussing the conditions in a Stern-Gerlach experiment. *J. von Neumann* (Princeton, N. J.).

Fahmy, M. A new form of the quantum equation. Philos. Mag. 28, 364–369 (1939). [MF 1120]

H. T. Flint [Proc. Roy. Soc. London, Ser. A. 150, 421–441 (1935)] has shown that by using the 5 dimensional unified field theory of Kaluza, the Dirac wave equation of the electron may be written in the simplified form $\gamma^\mu\partial\psi/\partial x^\mu=0$ (μ summed from 1 to 5), the electromagnetic terms of the Dirac equation being incorporated in the term in γ^5. Here the γ^μ are matrices which behave like a contravariant vector, and are related to the matrices α_x, α_y, α_z, α_m and α_0 of Dirac. They obey the commutation rules $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\gamma^{\mu\nu}$, where $\gamma^{\mu\nu}$ is the fundamental metric tensor of Kaluza which involves the electromagnetic potential vector in the terms $\gamma^{\mu5}$. Closely related to γ^μ is the matrix $K_\nu{}^\mu=\partial\gamma^\mu/\partial x^\nu+\Gamma_{\nu\lambda}{}^\mu\gamma^\lambda$ (where $\Gamma_{\nu\lambda}{}^\mu$ is the Christoffel symbol), which obeys the quantum law $K_\mu{}^\mu=0$. Since the commutation rules hold if $K_\nu{}^\mu$ has the form $\gamma^\mu\Gamma_\nu-\Gamma_\nu\gamma^\mu$, where Γ_ν is any covariant matrix, the author seeks a suitable form for Γ_ν. It is shown that a very reasonable assumption leads to the formula

$$\Gamma_\nu=-\frac{\gamma^\nu}{\gamma^{\nu\nu}}\sum_{\mu\neq\nu}\left(\gamma^\mu\frac{\partial}{\partial x^\mu}\right),\qquad \nu \text{ not summed},$$

which expresses Γ_ν as an operator. With this definition of Γ_ν, the quantum law $K_\mu{}^\mu=0$ holds as an operator equation. *O. Frink* (State College, Pa.).

Yamamoto, Hideo. On the gravitational perturbation for the Dirac electron. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. 22, 225–235 (1939). [MF 1492]

The basis of this paper is the tensorial generalization of the Dirac equation given previously by the author [Jap. J. Phys. 11, 35 (1936); 12, 27 (1938)], namely

$$(\hbar/i)J^{\sigma\kappa}{}_{\cdot\cdot\lambda\nu}(\nabla_\sigma+ie\phi_\sigma/\hbar c)\psi^{\lambda\nu}+mc\psi^\kappa=0,$$

$$(\hbar/i)J^{\sigma\cdot\lambda\nu}_{\cdot\kappa}(\nabla_\sigma+ie\phi_\sigma/\hbar c)\psi^\kappa+mc\psi^{\lambda\nu}=0,$$

$$J_{\sigma\kappa\lambda\nu}=\tfrac{1}{2}(g_{\sigma\kappa}g_{\lambda\nu}+g_{\sigma\lambda}g_{\kappa\nu}-g_{\sigma\nu}g_{\kappa\lambda}+i(-g)^{\frac{1}{2}}\epsilon_{\sigma\kappa\lambda\nu}),$$

where ϕ_σ is the electromagnetic 4-potential and ∇_σ the operator of covariant differentiation. A method of dealing with perturbations is developed, and applied to the case where the perturbation on a central electrostatic field is a weak gravitational field corresponding to $g_{11}=g_{22}=g_{33}=-(1+2\gamma\Phi)$, $g_{44}=(1-2\gamma\Phi)$, $g_{\mu\nu}=0$ $(\mu\neq\nu)$; γ is a small constant and Φ a general function of the radius vector. *J. L. Synge.*

Fierz, M. and Pauli, W. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. Proc. Roy. Soc. London, Ser. A. 173, 211–232 (1939). [MF 847]

In this paper the authors generalize the work of one of

them [Fierz, Helvetica Phys. Acta **12**, 3 (1939)] and obtain equations for particles of spin greater than one in the presence of an external electromagnetic field. Since it is impossible to obtain a consistent theory by replacing the operator $\partial/\partial x^\nu$ in Fierz' equations by $\partial/\partial x^\nu-(ie/\hbar c)\varphi_\nu$, they are forced to use an artifice; they set up a Lagrangian function which contains the field quantities describing the particle, supplementary fields which describe particles of lower spin, and the four vector potentials of the external field. The supplementary fields are introduced in the Lagrangian in such a manner that when no external field is present the supplementary fields and their time derivatives vanish as a consequence of the equations obtained by the variation of the Lagrangian function. These equations together with others obtained from the variation equations are called supplementary conditions. It is shown that the number of supplementary conditions remains the same when an external electromagnetic field is introduced. However, the role played by the particles described by the supplementary fields is not discussed when an external field is present. The Lagrangian function is not uniquely determined for the cases of higher spin. The cases for spin 2 and 3/2 are treated in detail and it is further shown that the equations obtained in the former case go over to those of the relativity theory of weak gravitational fields when the rest mass vanishes and there is no external electromagnetic field. *A. H. Taub* (Seattle, Wash.).

Schönberg, Mario. Équations relativistes de mouvement du premier ordre en Mécanique quantique. C. R. Acad. Sci. Paris **209**, 985–987 (1939). [MF 1200]

The author investigates the problem of linearization of the Gordon-Klein relativistic equation and finds a new form of spinor equations besides that formulated previously by Dirac [Proc. Roy. Soc. London, Ser. A. **155**, 447 (1936)]. An expression for a density vector and a Lagrangian for this new form of equations are included. *L. Infeld.*

Roubaud-Valette, Jean. La mécanique ondulatoire de certains espaces tordus. C. R. Acad. Sci. Paris **210**, 168–169 (1940). [MF 1631]

Pastori, Maria. I principali invarianti del campo elettromagnetico in teoria della relatività. Ist. Lombardo, Rend. **72**, 179–186 (1939). [MF 950]

Interpretations in terms of the electric and magnetic vectors of invariants formed from the electromagnetic tensor. *J. L. Synge* (Toronto, Ont.).

Matossi, F. Bemerkungen zur Analogie der Schrödingerschen Differentialgleichung mit einer Wellengleichung. Phys. Z. **41**, 47–52 (1940). [MF 1303]

Petiau, Gérard. Sur la représentation de l'équation d'ondes et l'évolution des grandeurs électromagnétiques dans la théorie du photon. J. Phys. Radium **10**, 413–419 (1939). [MF 1032]

Beim Lesen der Arbeiten der Schule von de Broglie zu der auch die vorliegende Arbeit gehört, ist zu beachten, dass diese Schule bei Untersuchung der mathematischen Gleichungen eine eigene physikalische Terminologie verwendet, welche zurückgeht auf die ältere Idee von de Broglie, dass das Photon eine von Null verschiedene Ruhmasse besitze. Deshalbe wurde von de Broglie vorgeschlagen [Nouvelles recherches sur la lumière, Paris, 1936], das Photon durch die Gleichungen

$$(1)\qquad \frac{1}{i}\frac{\partial\varphi}{\partial t}=\sum_{\nu=1}^{3}A_\nu\frac{\partial\varphi}{\partial x_\nu}+\lambda A_4\varphi=(\underline{H}\varphi),$$

mit den Nebenbedingungen

$$(2)\qquad 0=\sum_{\nu=1}^{3}B_\nu\frac{\partial\varphi}{\partial x_\nu}+\lambda B_4\varphi=0$$

zu beschreiben. Hierin hat die Wellenfunktion φ 16 Komponenten φ_{ik} $(i, k=1, \cdots, 4)$, die sich bei Lorentz-transformationen wie das direkte Produkt aus zwei Dirac'schen Wellenfunktionen transformieren ("Undor zweiter Stufe" nach Belinfante). Sind $\alpha_i^{(1)}, \alpha_r^{(2)}$ zwei miteinander vertauschbare Systeme Dirac'scher Matrices, die auf den ersten, beziehungsweise zweiten, Index von φ wirken, so wird für A_μ, B_μ $(\mu=1, \cdots, 4)$ angesetzt

$$(3)\qquad A_\mu=\tfrac{1}{2}(\alpha_\mu^{(1)}+\alpha_\mu^{(2)}),\quad B_\mu=\tfrac{1}{2}(\alpha_\mu^{(1)}-\alpha_\mu^{(2)}).$$

Später sind dieselben Gleichungen von anderen Autoren [F. T. Belinfante, Physica **6**, 849, 870 (1939); N. Kemmer, Proc. Roy. Soc. London. Ser. A. **173** (1939)] in der Theorie des Mesons herangezogen worden. Diese Gleichungen beschreiben ein Teilchen mit Spin 1 und ein Teilchen mit Spin 0. Die de Broglie'sche Schule nennt die das Teilchen mit Spin 1, beziehungsweise das mit Spin 0, beschreiben den Grössen (Vektor und schiefer Tensor, beziehungsweise Pseudovektor und Pseudoskalar; ein weiterer Skalar verschein der vermöge (2)) "Maxwell'sche Grössen," beziehungsweise nicht-Maxwell'sche Grössen, und die ganzen Überlegungen "Theorie des Photons."

In der vorliegenden Note wird zunächst die allgemeine Lösung der Feldgleichungen für ebene Wellen bei gegebenen Anfangswerten der Feldgrössen diskutiert. Sodann werden die Eigenwerte der Operatoren

$$S_x=t\tfrac{1}{2}[i\alpha_2^{(1)}\alpha_0^{(1)}+i\alpha_2^{(2)}\alpha_3^{(2)}], \cdots$$

untersucht [über die Definition des Spins bei de Broglie vgl. T. Géhénian, L'electron et photon, Paris, 1938, insbesondere p. 92 ff. und 118 ff.] und gezeigt, dass für die Komponenten von $\vec{S}$ in der Richtung der Wellenfortpflanzung die Eigenwerte $+1$ und -1 zu Transversalschwingungen, der Eigenwert 0 zu Longitudinalschwingungen gehört. Die Betrachtungen werden weiter geführt in der nachstehend referierten Arbeit. *W. Pauli* (Zürich).

Petiau, Gérard. Sur la théorie générale des corpuscules élémentaires et la théorie du photon. J. Phys. Radium (7) **10**, 487–494 (1939). [MF 1420]

The author investigates the general principles of the theory of elementary particles with the aim of including de Broglie's theory of the photon into the framework of the general formalism. He obtains a formulation of this theory which does not make use of a special choice of matrices in the wave equations. The relativistic transformation properties of the fundamental expressions are discussed in detail. *V. F. Weisskopf* (Rochester, N. Y.).

Houston, W. V. Acceleration of electrons in a crystal lattice. Phys. Rev. **57**, 184–186 (1940). [MF 1101]

The motion of an electron in a periodic potential field (crystal lattice), and accelerated by a uniform field, can be obtained by treating the time-dependent Schrödinger-equation. Instead of the approximative solution of this equation suggested by the wave packet treatment [F. Bloch,

Z. Phys. 52, 555 (1928)], author assumes a solution which is an infinite series of such approximative solutions with time-dependent coefficients. For these time-dependent coefficients a system of ordinary linear differential equations of the first order is obtained. The result shows that the wave-vector increases linearly with the time within the bounds of a single Brillouin zone. At the boundaries of the zones transitions to other zones may take place if the accelerating field is large enough. *A. Erdélyi.*

✱De Donder, Th. L'énergétique déduite de la mécanique statistique générale. Chimie Math. 4, 76 pp. (1939).

In writing this paper the author has attempted by applying the principles of his Théorie nouvelle de la Mécanique statistique [Vol. I of Collection du Centre de Recherche] to obtain the fundamental laws of energetics and to develop their statistical significance. In Chapter I the transport equation is given in explicit form. Chapter II is devoted to the mechanics of continuous media following ideas connected with the equations of E. and F. Cosserat. In Chapter III the author studies the transport of total energy by each particle and obtains the first law of thermodynamics in a general form. In Chapter IV the balance in entropy in the second law on thermodynamics is studied. Chapter V is devoted to systems containing several constituents. In Chapter VI the thermodynamics in chemical systems not in equilibrium is studied as well as Gibbs' relation in chemical affinity. These are done both for homogeneous as well as heterogeneous systems. Finally a theory of reaction velocities is developed. *N. Wiener* (Cambridge, Mass.).

✱Fowler, R. H. and Guggenheim, E. A. Statistical Thermodynamics. Cambridge University Press, Cambridge, 1939. x+693 pp. $9.50.

This book may be described as a half brother of the recent enlarged second edition of R. H. Fowler's "Statistical Mechanics." Like the parent first edition, the latter discusses statistical methods from all angles, and includes chapters on astrophysical and atmospheric applications. The present work is intended for physicists and chemists, and the applications are confined to terrestrial physics and chemistry. It is mathematical, but omits many details of proofs given in Fowler's "Statistical Mechanics"; moreover, after it has derived the laws of thermodynamics from statistical mechanics, it makes use of thermodynamic reasoning as well as arguments of statistical mechanics, wherever this seems advantageous.

The first chapter describes the two fundamental assumptions of statistical mechanics, the atomic structure of matter and the rule for averaging; it has sections on assemblies and their states, on the accessibility of states for assemblies of similar systems, on symmetrical and anti-symmetrical types, on the eigenfunctions and on the enumeration of states and complexions. The arguments for the assumptions are not analyzed, reference for this being made to Tolman's "Foundations of Statistical Mechanics." The second chapter establishes the usual theorems of statistical mechanics for assemblies of permanent systems; it deals with the weights of the states of various types of oscillators and rotators and the enumeration of complexions of simple assemblies of systems, including degenerate systems; with the method of steepest descents, with average values and the statistical temperature; and with the relationship to thermodynamics. The remaining 90% of the book, in eleven chapters, applies the general theory to various particular types of systems in equilibrium, and in addition there is a chapter (XII) on chemical kinetics, which, though in a sense outside the scope of the book, shows how useful a tool the equilibrium theory has been in discussing rates of reaction; this chapter deals almost entirely with homogeneous gaseous reactions not depending on radiation.

The eleven equilibrium chapters deal with substances of not too great complication, for which partition functions based on particular molecular models can be constructed. The subjects are as follows: permanent perfect gases, perfect crystals, chemical equilibria and evaporation together with Nernst's theorem, grand partition functions and their applications (particularly to cooperative regular assemblies), imperfect gases (equation of state and condensation), liquids and solutions of non-electrolytes, solutions of electrolytes, surface layers, elementary electron theory of metals, lattice imperfections and order-disorder in crystals, and finally the electric and magnetic properties of matter. The content of physical and chemical phenomena discussed is extensive and the treatment lucid and masterly; the book gives an impressive view of the great range of conquest already to the credit of statistical mechanics in providing a theory of matter in bulk, though there remain important territories yet to conquer, particularly the liquid state. In the comparisons of the theory with experiment, the book is closely up to date, and the comparisons are illustrated by many diagrams.

A slight defect in a book so rich in content is that the index, which is a combined author and subject index, is rather inadequate in both these aspects, particularly the former. *S. Chapman* (London).

Fowler, R. H. and Guggenheim, E. A. Statistical thermodynamics of super-lattices. Proc. Roy. Soc. London, Ser. A. 174, 189–206 (1940). [MF 1287]

A "quasi-chemical" method for the treatment of order-disorder phenomena in alloys is developed, which is based on a formula strictly analogous to the law of mass action for gaseous chemical equilibria. It is shown that it corresponds to the solution of the statistical problem if the interdependence of permutation numbers for the different types of pairs of neighboring atoms is neglected. It is also shown that the new method is equivalent to Bethe's first approximation [H. A. Bethe, Proc. Roy. Soc. London, Ser. A. 150, 552] but easier to handle. It is applied to lattices of the *ab* type with equal and unequal numbers of the two sorts of atoms where it leads to already known results. Other applications are suggested. *L. W. Nordheim.*

Bieberbach, Ludwig. Über die Inhaltsgleichheit der Brillouinschen Zonen. Monatsh. Math. Phys. 48, 509–515 (1939). [MF 665]

The author proves a theorem which is of importance in the atomic theory of metals. This theorem is connected with 2 or more dimensional lattices in the following way. Join the point A_0 of the given lattice to all others A_n and erect the perpendicular bisectors on A_0A_n. A point P of the space belongs to the "first zone" if none of the perpendicular bisectors has an inner point in common with the interval A_0P; P belongs to the "nth zone" if exactly $n-1$ of the perpendicular bisectors have an inner point in common with A_0P. The successive zones are split up into an increasing number of small pieces, but the sum of these is stated by Brillouin to be constant. This is established by considering the group of translations of the space which leaves the lattice invariant. It is shown that each zone is a fundamental domain of this group. *O. Todd-Taussky.*

AUTHOR INDEX

PRINTED IN THE UNITED STATES OF AMERICA